Linear Quadratic Optimal Control Problem of Stochastic Switching Systems

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Abstract—**Switched systems have numerous applications in control of real systems as mechanical systems, automotive industry, aircraft and air traffic control, switching power converters, and many other fields. Optimal control problems of switching systems require the decision of both the optimal solutions and switching sequences. The stochastic optimal control problem of linear switching systems with quadratic cost function is investigated. The contribution of this paper is to present a necessary and sufficient condition of optimality for considered switching systems.**

Index Terms—**condition of optimality, quadratic function, optimal control, switching system**

I. INTRODUCTION

The optimal control problem for linear systems was solved, as well as the filtering one, in 1960s by Kalman [1], but there exist a lot of invariant and non-invariant linear systems with still open optimal control problem. There has been an enormously rich theory on LQ control, deterministic and stochastic alike (see [2]-[9]).

For the deterministic Riccati equation was essentially solved by Wonham [10] by applying Bellman's principle of quasilinearizatin (see [11]).

Bismut [12] performed a detailed analysis for stochastic LQ control with random coefficients. But associated Riccati equation in this case is a highly nonlinear backward stochastic differential equations. The existence and uniqueness of solution for such class of equations was investigated in [12]. Switching systems consist of several subsystems and a switching law indicating the active subsystem at each time instantly. For general theory of stochastic switching systems it is referred to [13] . Theoretical results and applications of stochastic switching systems were developed in [14]-[18]. Deterministic and stochastic optimal control problems of switching systems are actual at present [19]-[22].

This article is concerned with optimal control problem of stochastic linear switching systems with quadratic cost functional.

The rest of paper is organized as follows. The next section formulates the main problem, presents some concepts and assumptions. The necessary and sufficient condition of optimality for stochastic linear switching systems is obtained in section III. In section IV, the unique optimal control is derived in terms of state feedback via the solution of Riccati equation. The paper is concluded in section V with some possible developments and enlargements.

II. NOTATIONS AND PROBLEM FORMULATION

In this section we fix notation and definition used throughout this paper. Let N be some positive constant, R^n denotes the n - dimensional real vector space, $|.|$ denotes the Euclidean norm and $\langle \cdot, \cdot \rangle$ denotes scalar product in R^n. E represents expectation ; the set of integer numbers $1,...,s$ is denoted by $\overline{1,s}$; ' (the prime) denotes derivative; * is the matrix transposition operation.

Let $w_t^1, w_t^2,...,w_t^s$ are independent Wiener processes, which generate filtrations $F_t^l = \overline{\sigma}(w_q^l, t_{l-1} \le t \le t_l), l = \overline{1,s}$; $(\Omega, F^l, P), l = \overline{1,s}$ be a probability spaces with corresponding filtrations $\left\{ F_t^l, t \in [t_{l-1}, t_l] \right\}$. $L_{F^l}^2 (a,b; R^n)$ denotes the space of all predictable processes $x(t,\omega) \equiv x_t(\omega)$ such that: $E \int_a^b |x_t(\omega)|^2 dt < +\infty$. $R^{m \times n}$ is the space of all linear transformations from R^m to R^n.

Let $O_l \subset R^{n_l}, Q_l \subset R^{m_l}$ be open sets; $T = [0,T]$ be a finite interval and $0 = t_0 < t_1 < ... < t_s = T$. Following notation is used unless specified otherwise: $\mathbf{t} = (t_0, t_1,...t_s), \mathbf{u} = (u^1, u^2,...u^s), \mathbf{x} = (x^1, x^2,...x^s)$. Consider following linear controlled system with variable structure:

$$dx_t^l = \left(A_t^l x_t^l + B_t^l u_t \right) dt + \left(C_t^l x_t^l + D_t^l u_t^l \right) dw_t^l \qquad t \in (t_{l-1}, t_l], \quad l = \overline{1,s} \tag{1}$$

$$x_{t_l}^{l+1} = \Phi_{t_l}^l x_{t_l}^l + K_{t_l}^l , l = \overline{1,s-1}; \quad x_{t_0}^1 = x_0 \tag{2}$$

$$u_t^l \in U_\partial^l \equiv \left\{ u^l (\cdot, \cdot) \in L_F^2 (t_{l-1}, t_l; R^m) | u^l (t, \cdot) \in U^l \subset R^m \right\} \tag{3}$$

where $U^l, l = \overline{1,r}$ are non-empty bounded sets. The elements of U_∂^l are called the admissible controls. The problem is concluded to find the optimal solution $(x^1, x^2, \ldots, x^s, u^1, u^2, \ldots u^s)$ and the switching sequence $t_1, t_2, \ldots t_s$ which minimize the cost functional :

$$J(u) = E\left\{\langle Gx_T^s, x_T^s \rangle + \sum_{l=1}^s \int_{t_{l-1}}^{t_l} \left(\langle M_t^l x_t^l, x_t^l \rangle + \langle N_t^l u_t^l, u_t^l \rangle \right) dt \right\} \quad (4)$$

Elements of matrices $A^l, B^l, C^l, M^l, N^l, \Phi^l$ and vectors $K^l, l = \overline{1,s}$ are continuous, bounded functions. G, $N^l, l = \overline{1,s}$ are a positively semi-defined matrices , and $N^l, l = \overline{1,s}$ are a positively defined matrices.

Let $U = U^1 \times U^2 \times \ldots \times U^s$, and consider the sets $A_i = T^{i+1} \times \prod_{j=1}^i O_j \times \prod_{j=1}^i U^j$ with the elements $\pi^i = (t_0, \ldots t_i, x_{t_1}^1, \ldots x_{t_i}^i, u^1, \ldots u^i)$.

Definition 1. The set of functions $\{x_t^l = x^l(t, \pi^l), t \in [t_{l-1}, t_l], l = \overline{1,s}\}$ is said to be a solution of the equation (1) with variable structure which correspond to an element $\pi^s \in A_s$, if the function $x_t^l \in O_l$ satisfies the conditions (2), while on the interval $[t_{l-1}, t_l]$ it is absolutely continuous a.c. (almost certainly) and satisfies the equation (1) a.e.(almost everywhere).

Definition 2. The element $\pi^s \in A_s$ is said to be admissible if the pairs $(x_t^l, u_t^l), t \in [t_{l-1}, t_l], l = \overline{1,s}$ are the solutions of system (1)-(3).

A_s^0 indicates the set of admissible elements.

Definition 3. The element $\tilde{\pi}^s \in A_s^0$, is said to be an optimal solution of problem (1)-(4) if there exist admissible controls $\tilde{u}_t^l, t \in [t_{l-1}, t_l], l = 1, \ldots s$ and corresponding solutions of system (1)-(2) such that pairs $(\tilde{x}_t^l, \tilde{u}_t^l), l = \overline{1,s}$ minimize the functional (4).

III. OPTIMAL CONTROL PROBLEM

Necessary and sufficient conditions satisfied by an optimal solution , play an important role for analysis of control problems. In this section optimality condition for stochastic control problem of linear switching systems is obtained.

Theorem. The necessary and sufficient conditions for an element $\pi^s = (t_0, \ldots t_r, x_t^1, \ldots, x_t^s, u^1, \ldots u^s)$ to be an optimal solution of problem (1)-(4) if and only if :

a) there exist random processes $(\psi_t^l, \beta_t^l) \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l}) \times L_{F^l}^2(t_{l-1}, t_l; R^{n_l \times n_l})$ which are the solutions of the following adjoint equations:

$$\begin{cases} d\psi_t^l = -\left[A_t^{l^*} \psi_t^l + C_t^{l^*} \beta_t^l - M_t x_t^l \right] dt + \beta_t^l dw_t, \\ t \in [t_{l-1}, t_l), \ l = \overline{1,s} \\ \psi_{t_l}^l = \psi_{t_l}^{l+1} \Phi_{t_l}^l, \ l = \overline{1, s-1} \\ \psi_T^s = -Gx_T^s; \end{cases} \quad (5)$$

b) the candidate optimal controls u_t^l are defined by:

$$N_t^{l^*} u_t^l = B_t^{l^*} \psi_t^l + D_t^{l^*} \beta_t^l, \ t \in [t_{l-1}, t_l], l = \overline{1,s}, \quad (6)$$

c) the transversality conditions

$$\psi_{t_l}^{l+1} (\Phi_{t_l}^{l'} x_{t_l}^l + K_{t_l}^{l'}) = 0, l = \overline{1, s-1} \quad (7)$$

hold.

Proof. Let u_t^l and $\bar{u}_t^l, l = \overline{1,s}$ be some admissible controls; call the vectors $\Delta \bar{u}_t^l = \bar{u}_t^l - u_t^l$ be an admissible increments of the controls u_t^l. By (1)-(2), the trajectories $x_t^l, \bar{x}_t^l, l = \overline{1,s}$ correspond to the controls $u_t^l, \bar{u}_t^l$.

Consider two sequence laws $\mathbf{t} = (0, t_1, t_2 .. t_{s-1}, T)$ and $\bar{\mathbf{t}} = (0, \bar{t}_1, \bar{t}_2 .. \bar{t}_{s-1}, T)$. Increment of cost functional (4) along admissible control $\bar{\mathbf{u}} = (\bar{u}_t^1, \bar{u}_t^2, .. \bar{u}_t^s)$ looks like:

$$\begin{aligned} \langle J'(\mathbf{u}), \bar{\mathbf{u}} - \mathbf{u} \rangle &= E\{\langle Gx_T^s, \bar{x}_T^s - x_T^s \rangle + \\ &\sum_{l=1}^s \int_{\bar{t}_{l-1}}^{\bar{t}_l} \left(\langle M_t^l x_t^l, \bar{x}_t^l - x_t^l \rangle + \langle N_t^l u_t^l, \bar{u}_t^l - u_t^l \rangle \right) dt \} \end{aligned} \quad (8)$$

Taking into consideration (1)-(2) we obtain:

$$\begin{cases} d(\bar{x}_t^l - x_t^l) = \left[A_t^l (\bar{x}_t^l - x_t^l) + B_t^l (\bar{u}_t^l - u_t^l) \right] dt + \\ \left[C_t^l (\bar{x}_t^l - x_t^l) + D_t^l (\bar{u}_t^l - u_t^l) \right] dw_t^l, \ t \in [\bar{t}_{l-1}, \bar{t}_l) \\ \Delta x_{t_l}^{l+1} = \bar{x}_{t_l}^{l+1} - x_{t_l}^{l+1} = \Phi_{t_l}^l \bar{x}_{t_l}^l - \Phi_{t_l}^l x_{t_l}^l \end{cases} \quad (9)$$

According to Ito's formula for each $t \in [t_{l-1}, t_l], l = \overline{1,s}$:

$$\begin{aligned} d\langle \psi_t^l, (\bar{x}_t^l - x_t^l) \Delta t_l \rangle &= \langle d\psi_t^l, (\bar{x}_t^l - x_t^l) \Delta t_l \rangle + \langle \psi_t^l, d(\bar{x}_t^l - x_t^l) \Delta t_l \rangle \\ &+ \langle \beta_t^l, C_t^l (\bar{x}_t^l - x_t^l) \Delta t_l + D_t^l (\bar{u}_t^l - u_t^l) \Delta t_l \rangle dt \end{aligned}$$

Integrating this equality and taking expectation of both side into account (9) as follows:

$$\begin{aligned} &E\langle \psi_{t_l}^l, (\bar{x}_{t_l}^l - x_{t_l}^l) \Delta t_l \rangle - E\langle \psi_{t_{l-1}}^l, (\bar{x}_{t_{l-1}}^l - x_{t_{l-1}}^l) \Delta t_{l-1} \rangle = \\ &E \int_{t_{l-1}}^{t_l} \langle d\psi_t^l + A_t^{l^*} \psi_t^l + C_t^{l^*} \beta_t^l, (\bar{x}_t^l - x_t^l) \Delta t_l \rangle + \\ &E \int_{t_{l-1}}^{t_l} \langle B_t^{l^*} \psi_t^l + D_t^{l^*} \beta_t^l, (\bar{u}_t - u_t) \Delta t_l \rangle dt \end{aligned}$$

Due to this equality the expression (8) can be rewritten as:

$$\begin{aligned} \langle J'(\mathbf{u}), \bar{\mathbf{u}} - \mathbf{u} \rangle &= E\langle Gx_T^s, \bar{x}_T^s - x_T^s \rangle + E\sum_{l=1}^s \langle \psi_{t_l}^l, (\bar{x}_{t_l}^l - x_{t_l}^l) \Delta t_l \rangle + \\ &E\sum_{l=1}^s \int_{t_{l-1}}^{t_l} \left(\langle M_t^l x_t^l, (\bar{x}_t^l - x_t^l) \Delta t_l \rangle + \langle N_t^l u_t^l, (\bar{u}_t^l - u_t^l) \Delta t_l \rangle \right) dt \end{aligned}$$

$$-\mathrm{E}\sum_{l=1}^{s}\int_{t_{l-1}}^{t_{l}}\langle d\psi_t^l + A_t^{l^*}\psi_t^l + C_t^{l^*}\beta_t^l, (\overline{x}_t^l - x_t^l)\Delta t_l\rangle -$$

$$E\sum_{l=1}^{s}\int_{t_{l-1}}^{t_{l}}\langle B_t^{l^*}\psi_t^l + D_t^{l^*}\beta_t^l, (\overline{u}_t^l - u_t^l)\Delta t_l\rangle dt \qquad (10)$$

The stochastic processes ψ_t^l, at the points $t_1, t_2, \ldots t_s$ can be defined as follows:

$$\psi_{t_l}^l = \psi_{t_l}^{l+1}\Phi_{t_l}^l, \ l = \overline{1, s-1} \ \text{ and } \ \psi_T^s = -Gx_T^s$$

Further using equation (5) we get:

$$\langle J'(\mathbf{u}), \overline{\mathbf{u}} - \mathbf{u}\rangle =$$

$$E\sum_{l=1}^{s}\int_{t_{l-1}}^{t_l}\langle N_t^l u_t^l - B_t^{l^*}\psi_t^l - D_t^{l^*}\beta_t^l, (\overline{u}_t^l - u_t^l)\Delta t_l\rangle dt \qquad (11)$$

It is well known that the necessary and sufficient condition of optimality for convex functional given by as: $J'(\mathbf{u}) = 0$. The validity of (6)-(8), follows from the relation (11). Finally, according to the independence of increments $\Delta\overline{x}_t^l$, $\Delta\overline{u}_t^l$ $\Delta\overline{t}_l$, sufficiency is proceed from expression (10).

IV. Riccati Equations

This section is devoted to the the Riccati equation for the possible feedback regulator design of stochastic LQ problem of switching systems.

$$d\psi_t^l = -p_t^l x_t^l \ , \ l = \overline{1, s} \ , \text{ a.c.} \qquad (12)$$

At the end are received the differential equations, which are a stochastic analogue of the Riccati equations, for determi- nation functions p_t^l, $l = \overline{1, s}$.

We will search for p_t^l, $l = \overline{1, s}$ in the following form:

$$dp_t^l = \alpha_t^l dt + \gamma_t^l dw_t \ , \ l = \overline{1, s}$$

According to formula Ito have:

$$d\psi_t = -\left[dp_t^l x_t + p_t^l dx_t + \gamma_t\left(C_t^l x_t + D_t^l u_t^l\right)dt\right].$$

Using (1) and (5) have:

$$\left[-A_t^{l^8}\psi_t^l - C_t^{l^8}\beta_t^l + M_t^l x_t^l\right]dt + \beta_t^l dw_t^l =$$
$$-\left[\alpha_t^l x_t^l dt + \gamma_t^l x_t^l dw_t^l + p_t^l A_t^l x_t^l dt + p_t^l B_t^l u_t^l dt +\right. \qquad (13)$$
$$\left. p_t^l\left(C_t^l x_t^l + D_t^l u_t^l\right)dw_t^l + \gamma_t^l\left(C_t^l x_t^l + D_t^l u_t^l\right)dt\right].$$

For β_t^l, $l = \overline{1, s}$ we are having next form:

$$\beta_t^l = -\left[\gamma_t^l x_t^l + p_t^l C_t^l x_t^l + p_t^l D_t^l u_t^l\right], t \in [t_{l-1}, t_l] \qquad (14)$$

By means of simple transformations into account (14) expression (13) can be rewritten as follows :

$$\left[\alpha_t^l + p_t^l A_t^l + A_t^{l^*}p_t^l + \gamma_t^l C_t^l + C_t^{l^*}\gamma_t^l + C_t^{l^*}p_t^l C_t^l + M_t\right]x_t^l dt$$
$$+ p_t^l B_t^l u_t^l dt + \gamma_t^l D_t^l u_t^l dt + C_t^{l^*}p_t^l D_t^l u_t^l dt = 0 . \qquad (15)$$

Considering (12) in expression (6) optimal control can be defined explicitly for each $l = \overline{1, s}$ and $t \in [t_{l-1}, t_l)$:

$$u_t^l = -\left[N_t + D_t^{l^*}p_t^l D_t^l\right]^{-1}\left[B_t^{l^*}p_t^l + D_t^{l^*}\gamma_t^l + D_t^{l^*}p_t^l C_t^l\right]x_t^l,$$

here random processes (p_t^l, γ_t^l) are the solutions of following Riccati equations:

$$dp_t^l = -\left[p_t^l A_t^l + A_t^{l^*}p_t^l + \gamma_t^l C_t^l + C_t^{l^*}\gamma_t^l + C_t^{l^*}p_t^l C_t^l + M_t\right.$$
$$-\left(p_t^l B_t^l + \gamma_t^l D_t^l + C_t^{l^*}p_t^l D_t^l\right)\left(N_t + D_t^{l^*}p_t^l D_t^l\right)^{-1}$$
$$\left(B_t^{l^*}p_t^l + D_t^{l^*}\gamma_t^l + D_t^{l^*}p_t^l C_t^l\right)\right]dt + \gamma_t^l dw_t^l .$$

V. Conclusion

This work deals with description the real phenomena with non-invariant nature and investigation of optimal control problems of such linear systems.The results can be used in various optimization problems of biology, physics, engineering, economics, and have a lot of life science, financial market applications [23-27]. The LQ problem considered in this manuscript can be viewed as development of the problems formulated in [17], 21], [28].

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