Robust Control of Double Inverted Pendulum System

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***Abstract*—Double inverted pendulum system is highly nonlinear and unstable system, thus its stability is a matter of concern, particularly when the system components have parametric uncertainty. The aim is to balance the two pendulums vertically on a movable cart. This paper presents first the dynamic modeling of the system based on Euler – Lagrangian method and then uncertain model is obtained by considering the parametric uncertainty in moment of inertia of pendulums and friction coefficient of hinges and cart. In this paper, reference trajectory control, disturbance rejection and robust performance using H_∞ and μ synthesis controllers are made. Both controller shows good transient response, disturbance rejection and robust stability, but μ synthesis controller provides the superior robust performance as compared to H_∞ controller.**

***Index Terms*—Double inverted pendulum system(DIPS), H_∞ Controller, Interconnected system, Linear Fractional Transformation (LFT), μ- Synthesis Controller**

I. Introduction

The double inverted pendulum is a highly nonlinear and unstable system. It also exhibits many problems found in industrial and robotic applications. Many modern technologies use the concept of inverted pendulum such as altitude control of space satellite and rockets, balancing of ships against tides etc. The aim is to balance the pendulums vertically on a movable cart. In the design of robust control system, it is conventionally assumed that the system is affected by structured and unstructured uncertainties. Thus the robust properties of closed loop system could be achieved by using a robust controller. Some significant results on robust control theory and some recent results on deterministic and probabilistic methods for systems with uncertainties has been reported [1] and [2]. Double inverted pendulum using H_∞ and μ- synthesis controllers has been designed with multiplicative output uncertainty and designed controller has been implemented on a microcomputer for laboratory experiments [3]. μ controlled system has a quite good performance and seems to be a little better than H_∞ controlled system with respect to both performance and robustness. To reduce the effect of dry friction between cart and rail for a single inverted pendulum using H_∞ controller, where dry friction is taken as disturbance input. The influence of friction can be reduced in the reference tracking problem with H_∞ control [4]. H_∞ loop shaping controller design for double inverted pendulum system has been reported [5]. Results shows that the designed controller work satisfactory. The relative stability and disturbance attenuation properties are investigated for a triple inverted pendulum, the controller design is based on H_∞ sub optimal control problem [6] and [7]. External disturbance torque has been incorporated as part of control problem. The effect of disturbance is minimized using H_∞ sub optimal control design. Several controllers like pole placement, LQR, H_∞ and μ are designed and compared with respect to their performance and robustness properties [8]. In which μ controller is found to have superior relative performance. H_∞ and μ synthesis controller are developed and their performance are compared for the two link rigid and flexible manipulators in which μ synthesis controller shown to have superior robust performance [9]-[11].

Since moment of inertia of the pendulums and viscous friction coefficient of hinges and cart are difficult to be estimated precisely. It makes sense to assume unknown deviations in these parameters. It would be important to treat uncertainties in such parameters as structured uncertainties' rather than congregate them as unstructured uncertainty.

In all the previous research work, reference trajectory control and disturbance rejection of double inverted pendulum system with parametric uncertainty in pendulum masses and viscous friction in the hinges and cart has not been considered. In this paper, the reference trajectory, disturbance rejection and robust performance using H_∞ and μ synthesis controllers have been compared. Both controllers show good robust stability.

II. Mathematical Modeling

A. Dynamic Modeling

The double inverted pendulum system consists of a cart placed on a track, and two aluminium arms

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connected to each other. These are constrained to rotate within a single plane. The axis of rotation is perpendicular to the direction of motion of the cart. The cart is moved by a servo motor. Fig. 1 shows the schematic diagram of double inverted pendulum system. All parameters and variables are defined in the Table I. The nominal values of parameters are given in Table II.

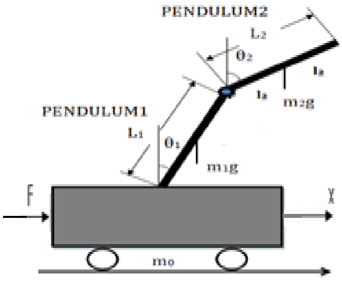

Figure 1. Schematic diagram of double inverted pendulum system.

TABLE I. SYSTEM NOMENCLATURE

Symbol	Description
m_1	Mass of the lower arm
m_2	Mass of the upper arm
m_0	Mass of the cart
L_1	Length of the lower arm
L_2	Length of the upper arm
l_1	Distance from bottom to centre of gravity of lower arm
l_2	Distance from bottom to centre of gravity of upper arm
c_0	Friction coefficient between cart and track
c_1	Friction coefficient between lower arm and cart
c_2	Friction coefficient between two arms
J_1	Inertia of lower arm around centre of gravity
J_2	Inertia of upper arm around centre of gravity
Θ_1	Angle between vertical and lower arm
Θ_2	Angle between vertical and upper arm
x	Cart position
τ_1	Disturbance torque to the lower arm
τ_2	Disturbance torque to the upper arm
u	Input voltage to the motor
g	Acceleration of gravity
t_m	Control torque

TABLE II. NOMINAL VALUES OF THE PARAMETERS

symbol	Values
m_1	0.548kg
m_2	0.41kg
m_0	2.0kg
L_1	0.35m
L_2	0.25m
J_1	0.0547kg-m^2
J_2	0.0521kg-m^2
c_0	0.0654Nms
c_1	0.0232Nms
c_2	0.0088Nms

The dynamic modeling based on Euler lagrangian formulation is obtained from the lagrangian(L).

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right)-\frac{\partial L}{\partial q_i}=\tau_i \quad \text{For i = 1,2,...,n} \tag{1}$$

Euler- lagrangian equations can be modified considering friction in the joints and motor.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=u-c_0\dot{x} \tag{2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right)-\frac{\partial L}{\partial \theta_1}=-c_1\dot{\theta}_1+c_2(\dot{\theta}_2-\dot{\theta}_1) \tag{3}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)-\frac{\partial L}{\partial \theta_2}=-c_2(\dot{\theta}_2-\dot{\theta}_1) \tag{4}$$

After solving equations (2), (3) and (4), these can be written in vector matrix form.

$$M\ddot{Y}+C\dot{Y}+GY=F \tag{5}$$

$$M=\begin{bmatrix} m_0+m_1+m_2 & (m_1l_1+m_1L_1)\cos\theta_1 & m_2l_2\cos\theta_2 \\ (m_1l_1+m_1L_1)\cos\theta_1 & (J_1+m_1l_1^2+m_2L_1^2) & m_2L_1l_2\cos(\theta_1-\theta_2) \\ m_2l_2\cos\theta_2 & m_2L_1l_2\cos(\theta_1-\theta_2) & J_2+m_2l_2^2 \end{bmatrix}$$

$$C=\begin{bmatrix} c_0 & (-m_1l_1+m_2L_1)\sin\theta_1\dot{\theta}_1 & m_2l_2\sin\theta_2\dot{\theta}_2 \\ 0 & c_1+c_2 & m_2L_1l_2\sin(\theta_1-\theta_2)\dot{\theta}_2-c_2 \\ 0 & m_2L_1l_2\sin(\theta_1-\theta_2)\dot{\theta}_2-c_2 & c_2 \end{bmatrix}$$

$$G=\begin{bmatrix} 0 \\ -(m_1l_1+m_2L_1)g\sin\theta_1 \\ -m_2gl_2\sin\theta_2 \end{bmatrix}, Y=\begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix}, \ F=\begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix}$$

B. LFT Modeling

Considering the parametric uncertainty in moment of inertia and friction coefficients of double inverted pendulum system. The uncertainty in the moment of inertia and friction coefficients are represented as

$$J_i=\overline{J}_i(1+P_i\delta_{j_i}), c_i=\overline{c}_i(1+s_i\delta_{c_i}) \tag{6}$$

where $\overline{J}_i$ and $\overline{c}_i$ are the nominal values of the corresponding moment of inertia and friction coefficient respectively. P_i = 10% and s_i= 15% are the maximum relative uncertainty in each of them. Where

$$-1 \leq \ \delta_{j_i}, \ \delta_{c_i} \ \leq 1; \ j_i = 1,2; c_i = 0,1,2$$

The system block diagram with uncertain parameters are shown in the Fig. 2

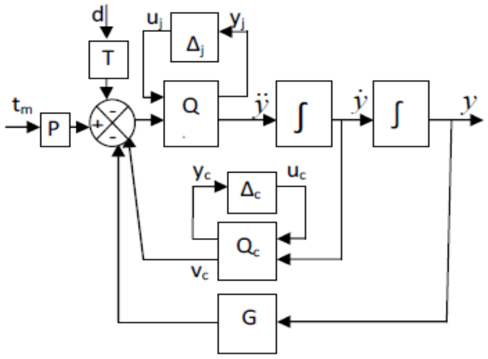

Figure 2. System block diagram with uncertain parameters.

After linearization of the model and choosing the state space variables as:

$$x_1 = x,\ x_2 = \theta_1,\ x_3 = \theta_2,\ x_4 = \dot{x},\ x_5 = \dot{\theta}_1,\ x_6 = \dot{\theta}_2 \tag{7}$$

The state and output equations are obtained as follows.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \multicolumn{3}{c}{\left[-\bar{M}^{-1}G\right]_{3X3}} & \multicolumn{3}{c}{\left[-\bar{M}^{-1}\bar{C}\right]_{3X3}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \multicolumn{3}{c}{\left[-\bar{M}^{-1}M_p\right]} & \multicolumn{3}{c}{\left[-\bar{M}^{-1}N_1\right]} & \multicolumn{2}{c}{\left[-\bar{M}^{-1}T\right]} & \left[-\bar{M}^{-1}P\right] \end{bmatrix} \begin{bmatrix} u_j \\ u_c \\ d \\ t_m \end{bmatrix}$$

$$\begin{bmatrix} y_j \\ y_c \\ y \end{bmatrix} = \begin{bmatrix} \multicolumn{3}{c}{\left[-\bar{M}^{-1}G\right]} & \multicolumn{3}{c}{\left[-\bar{M}^{-1}\bar{C}\right]} \\ \multicolumn{3}{c}{O_{3X3}} & \multicolumn{3}{c}{N_2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} +$$

$$\begin{bmatrix} \left[-\bar{M}^{-1}M_P\right] & \left[-\bar{M}^{-1}N_1\right] & \left[-\bar{M}^{-1}T\right] & \left[\bar{M}^{-1}P\right] \\ O_{3X3} & O_{3X3} & O_{3X2} & O_{3X1} \\ O_{3X3} & O_{3X3} & O_{3X2} & O_{3X1} \end{bmatrix} \begin{bmatrix} u_j \\ u_c \\ d \\ t_m \end{bmatrix}$$

The uncertain model of the whole system can be described by an upper LFT representation as shown in the Fig. 3.

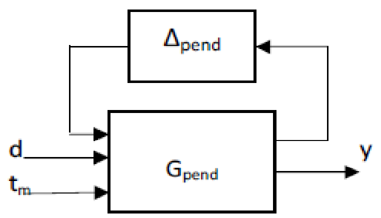

Figure 3. LFT model of double inverted pendulum system.

Thus the open loop double inverted pendulum system is a nine-input and nine-output system.

$$\begin{bmatrix} y_j \\ y_c \\ y \end{bmatrix} = G_{pend} \begin{bmatrix} u_j \\ u_c \\ d \\ t_m \end{bmatrix}$$

where G_{susp} and Δ_{susp} are defined as.

$$G_{pend} = \begin{bmatrix} [A]_{6x6} & [B_1]_{6x6} & [B_2]_{6x3} \\ [C_1]_{6x6} & [D_{11}]_{6x6} & [D_{12}]_{6x3} \\ [C_2]_{3x6} & [D_{21}]_{3x6} & [D_{22}]_{3x3} \end{bmatrix}$$

$$\Delta_{pend} = \begin{bmatrix} \Delta_j & 0 \\ 0 & \Delta_c \end{bmatrix}$$

C. Open loop Interconnected System

The structure of interconnected system is as shown in the Fig. 4. In order to achieve better performance two degree of freedom (2DOF) configuration is used. The feedback controller uses outputs x, θ_1 and θ_2 of double inverted pendulum to compute the control (u) driving the actuator. There are two external sources of disturbance acting on the two pendulums. Sensor noise acting on the three output measurement, modeled as weighting function w_n. Performance weights and control weights are w_p and w_u. The control objective can be interpreted to minimize the impact of disturbance inputs d_1 and d_2, on the outputs θ_1 and θ_2. Various weighting matrices are chosen as.

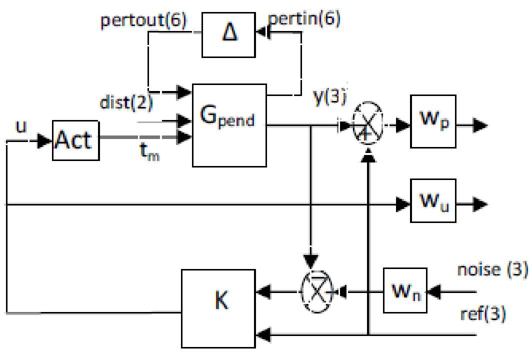

Figure 4. Structure of the interconnected system.

$$W_p = \begin{bmatrix} w_{p_1}(s) & 0 & 0 \\ 0 & w_{p_2}(s) & 0 \\ 0 & 0 & w_{p_3}(s) \end{bmatrix}; w_{p_1}(s) = \frac{s+10}{100s+1},$$

$$w_{p_2}(s) = w_{p_3}(s) = \frac{s+5}{s+100}$$

$$W_u = 10^{-3}$$

$$W_n = \begin{bmatrix} w_n(s) & 0 & 0 \\ 0 & w_n(s) & 0 \\ 0 & 0 & w_n(s) \end{bmatrix};$$

$$w_n(s) = 2X10^{-5}\frac{10s+1}{0.1s+1}$$

III. Robust Controller Design

A. H_∞ Controller

H_∞ optimization approach is an effective and efficient robust design method for linear, time invariant control systems. The robust design is to find a controller k for a given system such that, the closed loop system is robustly stable. For good tracking and disturbance attenuation, the design problem is to find a optimal controller which minimizes $\|(I+GK)^{-1}\|_\infty$ and for less control energy, $\|K(I+GK)^{-1}\|_\infty$ is to be minimized.

In order to have good tracking and disturbance rejection and to limit the control energy, we have to solve the mixed sensitivity problem. Its cost function can be described as

$$\left\| \begin{matrix} (I+GK)^{-1} \\ K(I+GK)^{-1} \end{matrix} \right\|_\infty \tag{8}$$

The above cost function may be recast into a standard H_∞ configuration shown in the Fig. 5.

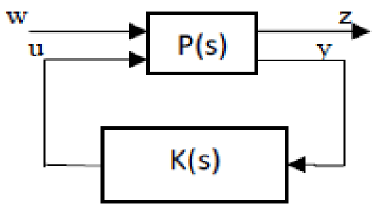

Figure 5. The standard H_∞ configuration.

P(s) is called the generalized plant/interconnected system. All the external inputs are denoted by w and z denotes the output signals to be minimized.

Y is the vector of measurements available to the controller. u is vector of control signals.

The objective is to find a stabilizing controller k to minimize the output z, in the sense of energy. Thus it is equivalent to minimize the H_∞ norm of the transfer function from w to z.

The design objective now becomes min $\|F(P,K)\|_\infty$, it is referred to as the H_∞ optimization problem.

B. μ Synthesis Controller

In standard M-Δ configuration as shown in the Fig. 6.

$$z = F_U(M,\Delta)w \text{ and } \|F_U(M,\Delta)\|_\infty \le 1 \tag{9}$$

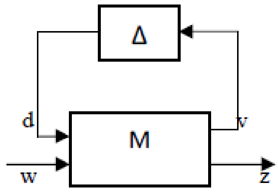

Figure 6. The standard M-Δ configuration.

It denotes the stability of $F_u(M,\Delta)$ which means the stability with respect to the plant perturbation Δ. The relation between M and p can be obtained by

$$M(p,k) = F_L(p,k) \tag{10}$$

For robust stability and robust performance, it is required to find a stabilizing controller k such that

$$\sup\ \mu\,[M(p,k)] \tag{11}$$

For optimal robust stability and robust performance, the objective is to solve for k such that

$$\text{Inf}\ \sup\ \mu[M(p,k)] \tag{12}$$

An iterative method is used to solve (12). The method is called D-K iteration synthesis method. It is based on solving the following optimization problem (13) for a stabilizing controller k and a diagonal constant scaling matrix D.

$$\text{Inf}\ \sup\ \inf \bar{\sigma}\,[DMD^{-1}(jw)] \tag{13}$$

IV. Simulation and Results

Double inverted pendulum system using robust controller has been designed in MATLAB. The closed loop transient and disturbance rejection responses have been obtained using H_∞ and μ- synthesis controllers. The transient response of the closed loop system is obtained using reference vector (measured in radians) given by $r = \begin{bmatrix} 1.0 \\ 0.2 \\ -0.2 \end{bmatrix}$, and disturbance vector (measured in N-m) is set to $d = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$

A. H_∞ Control

Fig. 7(a) and 7(b) show the closed loop transient responses of the system with H_∞ controller. The cart position has a settling time of 5s, whereas the lower and upper pendulums has returned to vertical position with a settling time of 6s.The response is fast with small overshoots of the output variables. The steady state errors are reduced to zero in all cases. These are very small as compared to the system without controller. Fig. 8 shows the closed loop disturbance rejection responses for the lower and upper pendulums. The effect of disturbance is minimized using H_∞ controller.

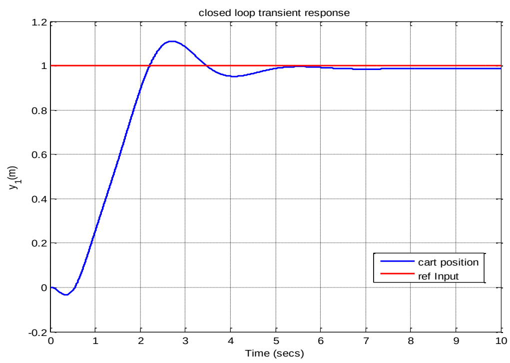

Figure 7 (a) Transient response (Cart Position).

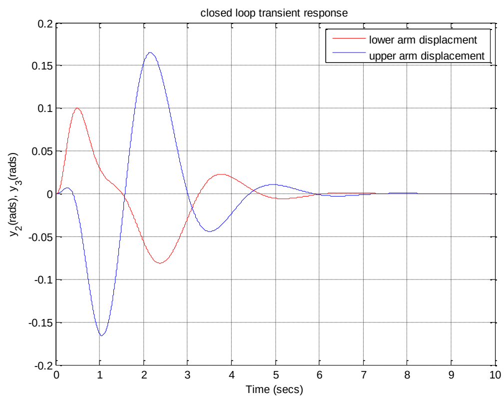

Figure 7 (b) Transient response (Lower and upper arm displacement).

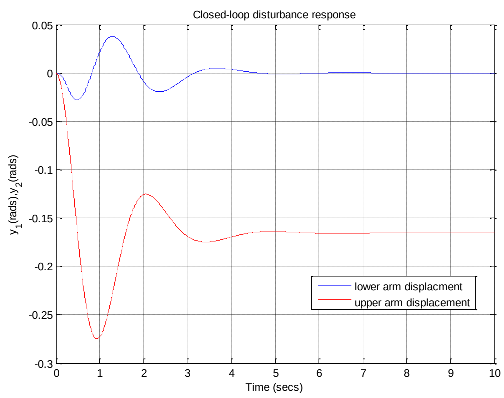

Figure 7. Disturbance rejection response.

Fig. 9(a) and 9(b) show the robust stability and robust performance of the system with H_∞ controller. The upper and lower bounds of the structured singular value (μ) are shown in the Fig. 9(a). It is clear that closed loop system with H_∞ controller achieve robust stability, since the maximum value of μ is 0.23. The μ value corresponding to robust performance analysis are shown in the Fig. 9(b).The closed loop system does not achieve robust performance, because the maximum value of μ is 1.10.Hence it is concluded that the designed H_∞ controller lead to good closed loop response, but doesn't ensure necessary robust performance of the closed loop system.

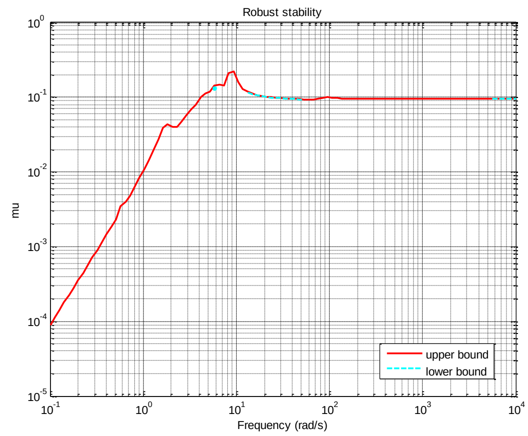

Figure 9(a) Robust stability.

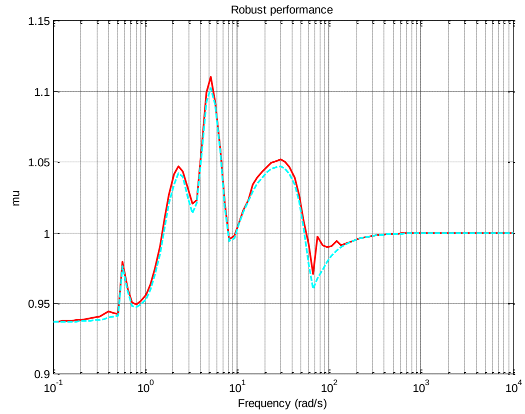

Figure 9(b) Robust performance.

B. μ -Synthesis Control

The closed loop transient response of the system with μ- synthesis controller has been shown in the Fig. 10(a) and 10(b). The response is fast with small overshoots of the output variables, but the response is slightly slower than those obtained with H_∞ controller. Reduction of steady state error is good. Fig. 11 shows the closed loop disturbance rejection responses for the lower and upper pendulums. With disturbance acting on the system, lower arm shows zero steady state error, but upper arm shows a steady state error of -0.16 radians.

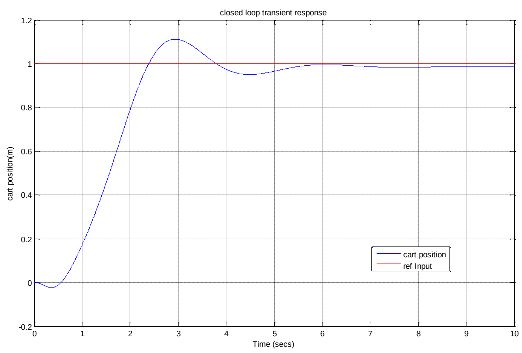

Figure 10 (a) Transient response(cart position).

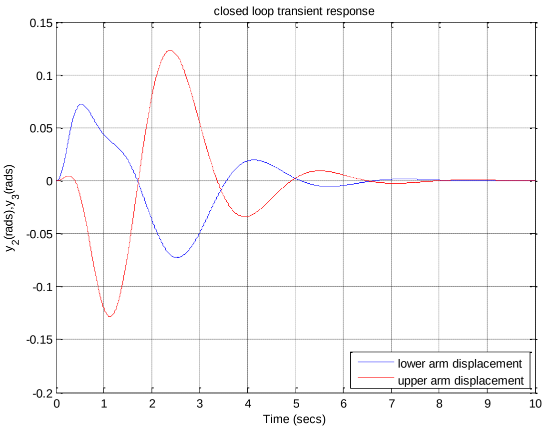

Figure 10(b) Transient response (Lower and upper arm displacement).

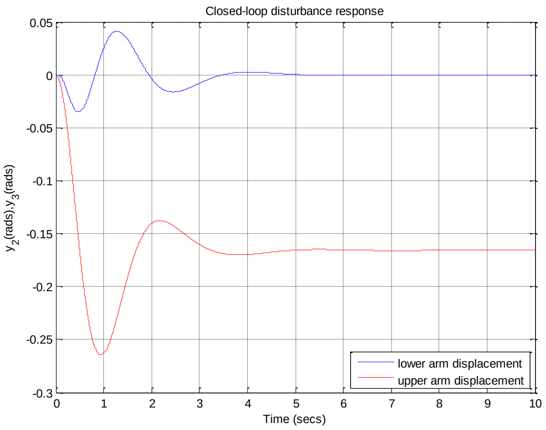

Figure 11. Disturbance rejection response.

Figs. 12 (a) and 12 (b) show the robust stability and robust performance of the system with µ- synthesis controller. The maximum value of structured singular value is 0.1, thus satisfying the criteria for robust stability. Fig. 12(b) shows that the maximum value of µ is less than one for frequencies less than 1KHZ, thus it ensures that good robust performance is achieved.

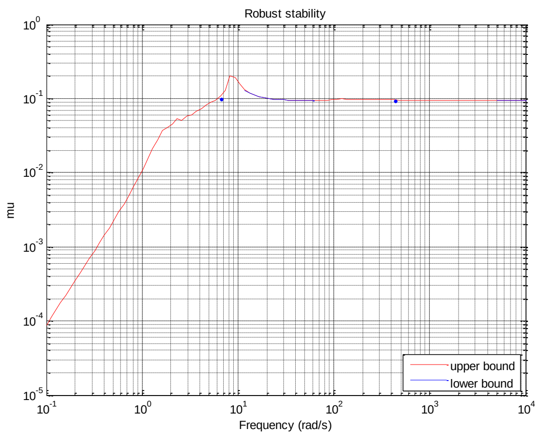

Figure 12 (a) Robust stability.

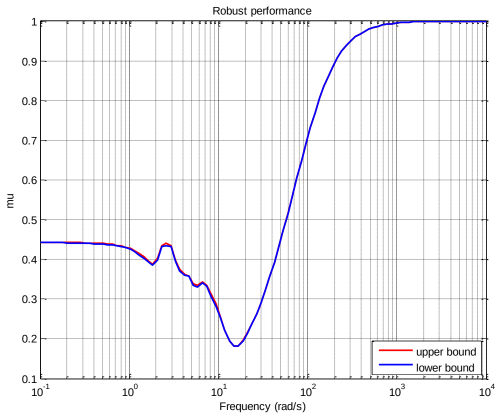

Figure 12 (b) Robust performance.

V. Conclusion

In this paper, H_∞ controller and µ- synthesis controllers are successfully designed using MATLAB for double inverted pendulum system. Both controllers are capable of stabilizing the system very effectively, show good reference trajectory control, disturbance rejection and robust stability. Both controllers ensure robust stability, but as far as robust performance is concerned, µ-synthesis controller provides the superior robust performance over the required frequency range.

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