Multiple Time Analysis of Weakly Coupled Non- linear Gyroscopic Systems

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Abstract— **In this paper, transient and steady state behavior of Weakly Coupled Non-Linear Gyroscopic Systems response, amplitude and frequency of oscillation as well as stability properties have been studied in detail. The method adopted for analysis is based on multiple time perturbation approach. The present system is a coupled two degree of freedom system with dissipative terms as well as terms corresponding to weakly exchanges of energy due to nonlinear terms.**

Index Terms—**synchronization, multiple time scaling, nonlinear coupled systems**

I. INTRODUCTION

For many applications in guidance and control it is necessary to have certain directional references. This reference serves as basis for obtaining navigational data or for stabilization of a vehicle or some of its equipments despite various inter-references rotatable with command. The device which has proved most suitable for the instrumentation of a reference direction is the Gyroscope. The motion of a rotating shaft with gyroscopic moments and non- linear restoring springs leads to a system of non-linear gyroscopic systems. We consider linear systems governed by equations having the form, where

$$[M]\ddot{u}+[G]\dot{u}+[A]u=0$$

u = column vector having n components

$[M]$ $= n\times n$ symmetric matrix

$[G]$ = $n\times n$ anti symmetric matrix

$[A]$ = $n\times n$ matrix

In this system $[M]\ddot{u}+[G]\dot{u}$ represents the inertia and $[G]\dot{u}$ represents the portion that is due to gyroscopic effects. Systems governed by equations containing inertial terms such as $[G]\dot{u}$ are called gyroscopic systems because their behavior is characteristic of gyroscope.

Systems of this type were analysed by a number of investigations Smith (1933), Tondl (1965), Black and Maternan (1968), Iwatsubo, Tomita and Kawai (1973) studied the vibrations of asymmetric shaft and rotor supported by a asymmetric bearing Fedorchenks (1958, 61) analyzed the motions of gyroscopes resting on vibrating supports. Danby (1964), Grevinikov (1964), Alfriend (1970), Nayfeh and Mamel (1970) and Nayfeh (1970a) analyzed the stability of triangular points in the elliptic restricted problems of three bodies. Balachandra(1976) presented a survey of non-linear gyroscopic systems. The motion of general gyroscopic systems was studied by Cherry (1924), Moser (1958), Bert (1961), Arnold (1963), Butenin (1965, Hori (1966), Alfriend and Richardson (1973), Balachandra (1973), Sethna and Balachandra (1974), Ranga-charyayulu and Srinivasan (1973), Bhansali and Thiruvenkatchar (1975).

Junkins, Jacobson and Blanton (1973) reduced the motion of any torque free rigid bodies to the solution of three uncoupled Duffing equations. Goodstain (1959) studied the free and forced vibrations of gyroscope. Throne (1961) solved numerically the equation governing the motion of gyroscopic under constant acceleration and correcting torque and Poli and Budrynes (1971) studied the symmetric gyroscope. Tiwari [1], [2] has considered higher order Non-linear system under free and forced condition and has established conditions for existence of limit cycle forced synchronized vibrations (both weak and strong forcing) for large number of classes of third order systems. Parametrically excited second orders as well as third order single degree of freedom systems have also been investigated for sustenance of Limit cycles. Singh [3] has considered the analysis of coupled nonlinear system under autonomous conditions using Multiple Time Scaling Technique. Linkein [4] have cited a very interesting example of Intestinal Coupled Vibrations and modeled the electrical activities through coupled Vanderpol oscillations. Various physical systems such as coupled mechanical system may also be modeled by two degree of freedom coupled systems. Nayfeh [5] has cited various practical vibrating systems and has used multiple time scaling technique. N. Sugiura, T. Hori and Y. Kawamura [6] provided a rationale for the emergence of synchronization in a system of coupled oscillators in a stick-slip motion. The single oscillator has a limit cycle in a region of the state space for each parameter set beyond the supercritical Hopf bifurcation. The two-oscillator system that has similar weakly coupled oscillators exhibit synchronization in a parameter range. However, it tends to be more in-phase for a non-identical

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pair with a rather weak coupling. Kre simir Josi ́c and Slaven Peles [7] present a general approach to the study of synchrony in networks of weakly nonlinear systems described by singularly perturbed equations. By performing a perturbative calculation based on normal form theory they analytically obtain an approximation to the Floquet multipliers that determine the stability of the synchronous solution. The technique allows proving and generalizing recent results obtained using heuristic approaches, as well as revealing the structure of the approximating equations.

This paper deals with the necessary conditions required for subharmonic and superharnonic synchronization in weakly non-liner systems with multiple inputs. For a second order system with a given non-linearity of all possible subharmonic, superharmonic and ultrasubharmonic synchronizations are determined. An example is furnished which illustrates the kind of possible synchronization. The stability of the synchronized solution has also been studied.

In [8] Phase models represent the ideal framework to investigate the synchronization of a non- linear oscillator with an external forcing. While many researchers focused the attention to their analysis, little work has been done about the reduction of a physical system to the corresponding phase model. In this paper we show how, resorting to averaging techniques, it is possible to obtain the phase model corresponding to a given set of state equations. As examples, we derive the phase equations and investigate the synchronization properties of two popular non-linaer oscillators.

There are a number of models of coupled oscillator networks where the question of the stability of fixed points reduces to calculating the index of a graph Laplacian. Some examples of such models include the Kuramoto and Kuramoto—Sakaguchi equations, which govern the behavior of generators coupled in an electric network. It can be shown that the index calculation can be related to a dual calculation which is done on the first homology group of the graph, rather than the vertex space and this representation is computationally attractive for relatively sparse graphs, where the dimension of the first homology group is low, as is true in many applications. It gives explicit formulae for the dimension of the unstable manifold to a phase-locked solution for graphs containing one or two loops. As an application, this presents some novel results for the Kuramoto model defined on a ring and compute the possible edge length for a stable solution.

In this paper, transient and steady state behavior of Weakly Coupled Non-Linear Gyroscopic Systems response, amplitude and frequency of oscillation as well as stability properties have been studied on the basis of Multiple Time Scaling perturbation approach.

II. Analysis

The system to be analyzed is:

$$\ddot{x}_1 + \mu\delta\dot{x}_1 + x_1 + \mu\beta x_1^2 x_2 = 0 \quad \ddot{x}_2 + \mu\eta\dot{x}_2 + x_2 + \mu\eta x_2^2 x_1 = 0 \qquad (1)$$

By two time technique, we know that

$$x_1(t,\tau) = x_{10}(t,\tau) + \mu x_{11}(t,\tau) \text{ +higher order terms} \qquad (2)$$

$$x_2(t,\tau) = x_{20}(t,\tau) + x_{21}(t,\tau) \text{ +higher order terms} \qquad (3)$$

where $\tau = \mu.t$, $t =$ fast time, $\tau =$ slow time

$$x_1^2 = x_{10}^2 + 2\mu x_{10}.x_{11}, \quad x_2^2 = x_{20}^2 + 2\mu x_{20}x_{21} \;. \qquad (4)$$

$$\dot{x}_1 = \frac{\delta x_{10}}{\delta t} + \mu\left(\frac{\delta x_{11}}{\delta t} + \frac{\delta x_{10}}{\delta \tau}\right), \quad \dot{x}_2 = \frac{\delta x_{20}}{\delta t} + \mu\left(\frac{\delta x_{21}}{\delta t} + \frac{\delta x_{20}}{\delta \tau}\right) \qquad (5)$$

$$\ddot{x}_1 = \frac{\partial^2 x_{10}}{\partial t^2} + \mu\left(\frac{\partial^2 x_{11}}{\partial t^2} 2\frac{\partial^2 x_{10}}{\partial t\partial \tau}\right), \quad \ddot{x}_2 = \frac{\partial^2 x_{20}}{\partial t^2} + \mu\left(\frac{\partial^2 x_{21}}{\partial t^2} + 2\frac{\partial^2 x_{20}}{\partial t\partial \tau}\right) \qquad (6)$$

Putting the values of (2), (3), (4), (5), (6) in (1)

$$\frac{\partial^2 x_{10}}{\partial t^2} + \mu\left(\frac{\partial^2 x_{11}}{\partial t^2} + 2\frac{\partial^2 x_{10}}{\partial t\partial \tau} + \right) + \mu\delta\left(\frac{\delta x_{10}}{\delta t} + \mu\left(\frac{\delta x_{11}}{\delta t} + \frac{\delta x_{10}}{\delta \tau}\right)\right)$$
$$+ (x_{10} + \mu x_{11}) + \mu\beta\left(x_{10}^2 + 2\mu x_{11}x_{10}\right)(x_{20} + \mu x_{21}) = 0 \qquad (7)$$

$$\frac{\partial^2 x_{20}}{\partial t^2} + \mu\left(\frac{\partial^2 x_{21}}{\partial t^2} + 2\frac{\partial^2 x_{20}}{\partial t\partial \tau}\right) + \mu\eta\left(\frac{\delta x_{20}}{\delta t} + \mu\left(\frac{\delta x_{21}}{\delta t} + \frac{\delta x_{20}}{\delta \tau}\right)\right)$$
$$+ (x_{20} + \mu x_{21}) + \mu\gamma\left(x_{20}^2 + 2\mu x_{20}x_{21}\right)(x_{10} + \mu x_{11}) = 0 \qquad (8)$$

Simplifying (7) and (8) and neglecting the terms of $o(\mu^2)$

$$\frac{\partial^2 x_{10}}{\partial t^2} + \mu\left(\frac{\partial^2 x_{11}}{\partial t^2} + 2\frac{\partial^2 x_{10}}{\partial t\partial \tau}\right) + \mu\delta\frac{\delta x_{10}}{\delta t} + (x_{10} + \mu x_{11}) + \mu\beta x_{10}^2 x_{20} = 0 \qquad (9)$$

$$\frac{\partial^2 x_{20}}{\partial t^2} + \mu\left(\frac{\partial^2 x_{21}}{\partial t^2} + 2\frac{\partial^2 x_{20}}{\partial t\partial \tau}\right) + \mu\eta\frac{\delta x_{20}}{\delta t} + (x_{20} + \mu x_{21}) + \mu\gamma x_{20}^2 x_{10} = 0 \qquad (10)$$

Comparing and selecting $o(\mu^0)$ and $o(\mu^1)$ terms from (9) and (10)

$$\frac{\partial^2 x_{10}}{\partial t^2}+x_{10}=0 \qquad \frac{\partial^2 x_{20}}{\partial t^2}+x_{20}=0 \tag{11}$$

$$\frac{\partial^2 x_{11}}{\partial t^2}+2\frac{\partial^2 x_{10}}{\partial t\partial \tau}+\delta\frac{\delta x_{10}}{\delta t}+x_{11}+\beta x_{10}^2x_{20}=0$$
$$\frac{\partial^2 x_{21}}{\partial t^2}+2\frac{\partial^2 x_{20}}{\partial t\partial \tau}+\eta\frac{\delta x_{20}}{\delta t}+x_{21}+\gamma x_{20}^2x_{10}=0 \tag{12}$$

(11) Has the solution at

$$x_{10}=R_1(\tau)\cos(\omega t+\phi_1(\tau)),\quad \omega=1\quad x_{20}=R_2(\tau)\cos(\omega t+\phi_2(\tau)) \tag{13}$$

(12) and (13) can be rewritten as:

$$\frac{\partial^2 x_{11}}{\partial t^2}+x_{11}=-2\frac{\partial^2 x_{10}}{\partial t\partial \tau}-\delta\frac{\delta x_{10}}{\delta t}-\beta x_{10}^2x_{20}$$
$$\frac{\partial^2 x_{21}}{\partial t^2}+x_{21}=-2\frac{\partial^2 x_{20}}{\partial t\partial \tau}-\eta\frac{\delta x_{20}}{\delta t}-\gamma x_{20}^2x_{10} \tag{14}$$

Differentiating (13) and putting in (14) and simplifying resulting equations:

$$\begin{aligned}\frac{\partial^2 x_{11}}{\partial t^2}+x_{11}&=2\dot{R}_1\sin(t+\phi_1)+2R_1\cos(t+\phi_1)\dot{\phi}_1+\delta R_1\sin(t+\phi_1)+\\&0.5\beta R_1^2R_2(\cos(t+\phi_1)\cos(\phi_1-\phi_2)-\sin(t+\phi_1)\sin(\phi_1-\phi_2))+\\&0.25\beta R_1^2R_2(\cos(3t+3\phi_1)\cos(\phi_1-\phi_2)+\sin(3t+3\phi_1)\sin(\phi_1-\phi_2))+\\&0.5\beta R_1^2R_2\cos(\phi_1-\phi_2+t)\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\partial^2 x_{21}}{\partial t^2}+x_{21}&=2\dot{R}_2\sin(t+\phi_2)+2R_2\cos(t+\phi_2)\dot{\phi}_2+\eta R_2\sin(t+\phi_2)+\\&0.5\gamma R_2^2R_1(\cos(t+\phi_2)\cos(\phi_1-\phi_2)-\sin(t+\phi_2)\sin(\phi_1-\phi_2))+\\&0.25\gamma R_2^2R_1(\cos(3t+3\phi_2)\cos(\phi_1-\phi_2)+\sin(3t+3\phi_2)\sin(\phi_1-\phi_2))+\\&0.5\gamma R_2^2R_1\cos(t+\phi_2-\phi_1)\end{aligned} \tag{16}$$

III. Seculer Terms

$$\frac{\delta R_1}{\delta\tau}+0.5\delta R_1-0.25\beta R_1^2R_2\sin(\phi_1-\phi_2)=0$$
$$R_1\frac{\delta\phi_1}{\delta\tau}+0.25\beta R_1^2R_2\cos(\phi_1-\phi_2)=0 \tag{17}$$

$$\frac{\delta R_2}{\delta\tau}+0.5\eta R_2-0.25\gamma R_2^2R_2\sin(\phi_1-\phi_2)=0$$
$$R_2\frac{\delta\phi_2}{\delta\tau}+0.25\gamma R_2^2R_1\cos(\phi_1-\phi_2)=0 \tag{18}$$

IV. Determination of Steady State Amplitude and Phase

Solving (17) and (18):

$$\frac{\delta R_1}{\delta\tau}=-0.5\delta R_1+0.25\beta R_1^2R_2\sin(\phi_1-\phi_2)$$
$$\frac{\delta R_2}{\delta\tau}=-0.5\eta R_2+0.25\gamma R_2^2R_2\sin(\phi_1-\phi_2) \tag{19}$$

$$\frac{\delta R_1}{\delta\tau}=f_1(R_1,R_2)=-0.5\delta R_1+0.25\beta R_1^2R_2$$
$$\frac{\delta R_2}{\delta\tau}=f_2(R_1,R_2)=-0.5\eta R_2+0.25\beta R_2^2R_1 \tag{20}$$

$$\frac{\delta f_1}{\delta R_1}=-0.5\delta+0.5\beta R_1R_2\quad \frac{\delta f_1}{\delta R_2}=0.5\beta R_1^2\quad \frac{\delta f_2}{\delta R_1}=-0.5\gamma R_2^2$$
$$\frac{\delta f_2}{\delta R_2}=-0.5\eta+0.5\beta R_2R_1 \tag{21}$$

$J=Jacobian\ matrix=\begin{bmatrix}\frac{\delta f_1}{\delta R_1} & \frac{\delta f_1}{\delta R_2}\\ \frac{\delta f_2}{\delta R_1} & \frac{\delta f_2}{\delta R_2}\end{bmatrix}$ $at\left(R_1^*,R_2^*\right)$. If we put

$R_1=R_2=0$, $J=\begin{bmatrix}-0.5\delta & 0\\ 0 & -0.5\eta\end{bmatrix}$

Substituting the values of $\delta=0.5,\eta=0.5$.

$$J=\begin{bmatrix}-0.25 & 0\\ 0 & -0.25\end{bmatrix}$$

Characteristics equation is $|\lambda I-J|=0$

$$\lambda^2+0.6\lambda+0.0875=0 \tag{22}$$

We know that if $\lambda^2+\alpha_2\lambda+\alpha_1=0$, the system will be stable if $\alpha_1\succ 0,\alpha_2\succ 0$

Solving $\alpha_1=0.0875\succ 0,\alpha_2=0.6\succ 0$, given Gyroscopic system is stable.

V. Simulation and Correspondence with Theory

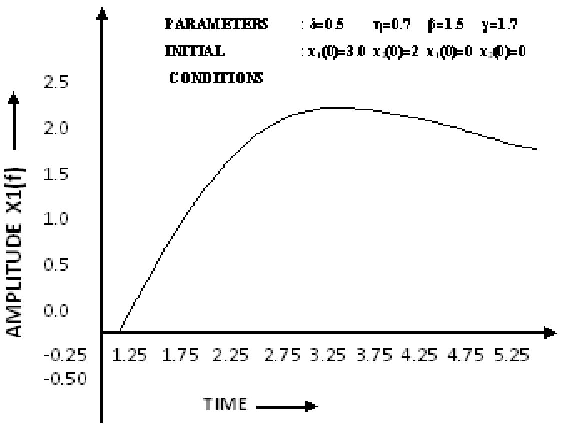

Figure 1. Simulation Result, Amplitude v/s Time

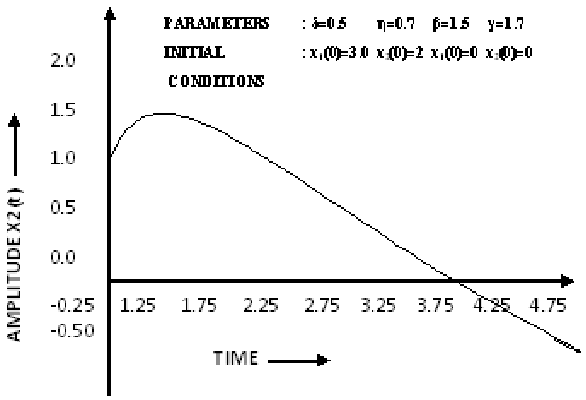

Figure 2. Simulation Result, Amplitude v/s Time

The present system is a coupled two degree of freedom system with dissipative terms as well as terms corresponding to weakly exchanges due to non-linear terms. The system is found to be slightly dependent upon initial conditions when the same parameters α, β are used. However, variation of parameters for one initial condition also gave significant change in response characteristics, it may be noted that β, γ, δ, η the system indicated is decaying tendency and the steady state is obtained to a zero value of amplitude of oscillations in both cases of $x_1(t), x_2(t)$. For one set of chosen parameters and initial conditions the response characteristics has been shown in Fig. (1) and Fig. (2). It is therefore expected that such systems where β, γ, δ, η are of order (1) multiple time scaling technique can be effectively employed.

VI. Conclusion

It has been demonstrated that the necessary and sufficient conditions for oscillations can be obtained by this technique. The order of accuracy in the proposed method is sufficient to predict magnitude of amplitude and phase as well as the stability properties. There is a close correspondence between the results theoretically predicted and simulation. Multiple time scaling technique is found to lead one more step forward for handling such systems. The transient and steady state behavior are well predicted and generation or quenching of oscillation can be nicely examined for dependence on various parameters. Amplitude v/s Time of considered gyroscopic systems are shown in following Figure (1) and Figure (2).

SIMULATION RESULTS

AMPLITUDE, $x_1(t)$

PARAMETERS: $\delta = 0.5, \eta = 0.7, \beta = 1.5, \gamma = 1.7$

INITIAL CONDITIONS:

$x_1(0) = 3.0,\ x_2(0) = 2.0,\ \dot{x}_1(0) = 0,\ \dot{x}_2(0) = 0$

AMPLITUDE, $x_2(t)$

PARAMETERS: $\delta = 0.5, \eta = 0.7, \beta = 1.5, \gamma = 1.7$

INITIAL CONDITIONS:

$x_1(0) = 3.0,\ x_2(0) = 2,\ \dot{x}_1(0) = 0,\ \dot{x}_2(0)$

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