A Synthesis Method of Robust Cascade Control System

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I. INTRODUCTION

The cascade control is one of the most popular structures to improve performance of the single-loop control systems, by that arose the cascade control system. In the published works [1] [2] [3] [4] [5] [6], the authors deal with various concepts of a cascade control system. However, there is no identical view of cascade control system among them. The authors give some cascade structure and refer to system with two control loops at the end. In [7], the author outlined a clear definition and most complete structure of a cascade control system which is a system having many closed-loops nested inside.

The most industrial control systems have to work in conditions when dynamic behaviour of object (process) often changes. At the same time, the disturbances occur out of rules and are difficult to determine. In such circumstance the problem synthesizing control system has to meet two principle requirements, as follows:

The system must have maximum stability margin to anticipate the uncertainty of the object.

System output is close to the set point as precisely as possible, under the action of disturbances.

The researchers of [1]-[6], for tuning cascade controllers, have not completely solved the robust stability problem of the system and just stop at cascade structure with two loops. This article will propose a new synthesis method based directly on the robust conditions of the system [7].

II. BACKGROUND THEORY

A. The Viewpoint of High-Robust Control System [7]

Let a control system has the transfer function:

\[ W(s) = \frac{A(s)}{D(s)} \]

where, \( s \) – complex variable; \( D(s) \) – the characteristic polynomial with \( q \) real roots: \( s = -\beta_k \pm j\alpha_k \), \( \alpha_k \geq 0 \), \( j = \sqrt{-1} \); \( q + 2p = n \) – the order of \( D(s) \). Thereafter, we call roots of the polynomial simply characteristic roots. Let acting step input with Laplace transform: \( z(s) = 1/s \).

Our purpose is to construct robust control system therefore we consider only stable ones. So, the output system response can be written in the form:

\[ y(t) = c_0 + \sum_{i=1}^{q} c_i e^{-\beta_{i}t} + \sum_{k=q+1}^{q+p} c_k e^{-\beta_{k}t} \sin(\alpha_k t + \phi_k) \]  

(1)

where, \( c_{ik} = 0,1, \ldots, q^{(p)} \) – constants; \( \alpha_k \) – frequency; \( \phi_k \) – phase of the oscillation; \( t \) – time variable.

We consider the oscillating component of (1):

\[ y_k(t) = c_k e^{-\beta_{k}t} \sin(\alpha_k t + \phi_k) \]  

(2)

The oscillation decay of process \( y_k(t) \) is defined as:

\[ d_k = \frac{y_k(t+T)}{y_k(t)} \]  

where \( T = 2\pi/\alpha_k \) – period. Substituting \( y_k(t) \) from (2), it’s easy to come to the relationship:

\[ d_k = e^{-2\alpha_k T} \]  

where \( m_k = 2\alpha_k \).

Clearly that the factor \( m_k \) determines oscillation behaviour of the \( y_k(t) \)-component (therefore in [8] it’s named “oscillation index”). Further, the greater \( m_k \) corresponds smaller decay \( d_k \) and less oscillation of the component. In the ultimate case, if \( m_k = +\infty \) then \( y_k(t) \) will not be oscillatory (\( d_k = 0 \)).

So, the system is more robust (more stable) if all the \( m_k \) are less oscillatory. Consequently, the system will be most robust, if all \( y_k(t) \) are not oscillatory, according to ultimate \( m_k = +\infty \).
In last case all characteristic roots trend into negative real and the system becomes a pure inertial (a lag system without oscilation). Among pure inertial systems, the simplest and most reliable is the first-order one.

Such, the most robust (high-robust) system has the transfer function:

$$W(s) = \frac{K}{1 + \theta s}, \quad \theta > 0$$ (4)

where, $K$ - transfer coefficient; $\theta$ - inertial constant.

**B. High Performance of High-Robust System [7]**

As shown in [7], a system under structure (4) will have best performance, when coefficient $K = 1$ and inertial constant $\theta$ is small as possible.

So, the high-robust system with high (best) performance has the transfer function:

$$W(s) = \frac{1}{1 + \theta s}, \quad \theta > 0$$ (5)

where, $\theta$ - inertial constant (the lag of desired system).

We call the system as high-quality system. Suppose, it is constructed as a single loop shown in Fig. 1, where, $z$ - reference; $y$ - output; $d$ - disturbance; $O(s), R(s)$ - the transfer functions of object and controller.

![Figure 1. Generic block diagram of a single-loop control system](image)

According to the diagram and formula (5), we have transfer function of the open-loop:

$$H(s) = \frac{W(s)}{1 - W(s)} = \frac{1/(1 + \theta s)}{1 - 1/(1 + \theta s)} = \frac{1}{\theta s}$$ (6)

And the controller:

$$R(s) = H(s)O(s) = \frac{1}{\theta s} O(s)^{-1}$$ (7)

It is there clear that, if the object is known, there is only parameter $\theta$ needed to determine for defining regulator.

**III. SYNTHESIS OF ROBUST CASCADE SYSTEM**

The problem synthesising cascade control system, is implies to ensure high-robust stability and high-performance of the system. It is stated as follows: Assuming that the control object changes uncertainly, the disturbance is arbitrary. Let’s determine controllers so that the stability margin of system is the largest and the control error is smallest.

**A. Transfer Function of the Open-Loop Cascade System**

As defined in [7], the cascade control system has many nested loops, in which each loop contains a cascade subsystem and its own dynamic blocks (Fig. 2).

![Figure 2. Cascade control scheme](image)

The notes in the Fig. 2 are: $z, y, u$ – respectively reference, output and control signal; $F_i, O_i, R_i$ – own dynamic blocks of $i$-th loop.

We take equivalent transformation of the diagram in Fig. 2 to single loop as in Fig. 3, where:

- $W_k$ – transfer function of subsystem, counting from the output of $R_k$ to input of $O_k$;
- $O_{Tk}$ – equivalent transfer function of part structure, counting from the output of $O_k$ to input of $R_k$.

![Figure 3. The equivalent single loop related to $R_k$ controller](image)

According to the Masson rule we have:

$$W_k = \frac{\prod_{j=1}^{n} R_j O_j}{1 + \sum_{j=1}^{n} F_j \prod_{i=j+1}^{n} R_i O_i}, \quad O_{Tk} = \frac{\sum_{j=1}^{n} F_j \prod_{i=j+1}^{n} R_i O_i}{O_j R_k}$$

From here, the transfer function of $k$-th open-loop (break out at the input of $R_k$) is defined as below:

$$H_k = R_k W_k O_k O_{Tk} = \frac{\sum_{j=1}^{n} F_j \prod_{i=j+1}^{n} R_i O_i}{1 + \sum_{j=1}^{n} F_j \prod_{i=j+1}^{n} R_i O_i}$$ (8)
B. Robust Cascade Controllers Synthesis

A cascade control system is high-quality one if the high-quality condition is satisfied for each equivalent single loop. Thus, by (4) each equivalent single open-loop satisfies following condition:

\[ H_k = \frac{1}{\theta_k s} \quad (9) \]

where, \( H_k \) - the open-loop transfer function of the \( k \)-th loop; \( \theta_k \) - the inertial constant of the loop.

Substituting (9) into (8) for all loops, we have following equations to find controllers (assume all \( F_k = 1 \) without loss of generality):

\[
\begin{align*}
\sum_{j=1}^{n} \prod_{j<i} R_{O_i} & = \frac{1}{\theta_k s}, \quad k = 1, n
\end{align*}
\]

\[ (10) \]

\[
\begin{align*}
\sum_{j=1}^{n} \prod_{j<i} R_{O_i} & = H_k, \quad k = 1, n
\end{align*}
\]

\[ (11) \]

Let’s note: \( x_j = \prod_{i=j}^{n} R_{O_i}, \quad \text{j} \leq n, \quad x_{n+1} = 1. \)

The system of equations (9) will become:

\[
\begin{align*}
\sum_{j=1}^{n} x_j & = H_k, \quad k = 1, n
\end{align*}
\]

\[ (12) \]

\[
\begin{align*}
H_{k+1} x_1 - x_2 & = 0, \\
H_{k+2} x_2 - x_3 & = 0, \\
& \quad \vdots \\
H_{k+n} x_n & = 0
\end{align*}
\]

\[ (13) \]

Solving the linear equations related \( x_i \) (\( i = 1, n \)), we receive controllers:

\[
R_k = \frac{x_k}{x_{k+1}} O_{i+1}, \quad k = 1, n
\]

\[ (14) \]

From here the controllers may be determined by following steps.

**Step 1**: Determine \( x_n \)

Multiply \( H_{n+1} \) and \( H_{n+1}^{-1} \) turn in the last two equations of (13), then subtract down and solve:

\[ H_{n+1}^{-1} H_{n+1} - H_{n+1}^{-1} x_n + H_{n+1}^{-1} x_n = H_{n+1}^{-1} - H_{n+1}^{-1} \]

\[ \Rightarrow x_n = H_{n+1}^{-1} - H_{n+1}^{-1} \]

\[ (15) \]

**Step 2**: Determine \( \frac{x_k}{x_{k+1}} \)

Subtract down two consecutive equations of (13) and solve, we have:

\[ \frac{x_k}{x_{k+1}} = H_{k+1}^{-1} - H_{k+1}^{-1} \sum_{i=1}^{k} \]

\[ \text{With } k = 1: \]

\[ \frac{x_1}{x_{2}} = H_{2}^{-1} + 1 \frac{H_{2}^{-1} - H_{2}^{-1}}{H_{2}^{-1} - H_{2}^{-1}} \]

\[ (16) \]

\[ \text{With } k = 2, n: \]

\[ \frac{x_k}{x_{k+1}} = \frac{(H_{k+1}^{-1} + 1)(H_{k+1}^{-1} - H_{k+1}^{-1})}{H_{k+1}^{-1} - H_{k+1}^{-1}} \]

\[ (17) \]

Replace \( H_k \) in the equations (15), (16), (17) by expression in (9), we have the following results:

\[ \frac{x_n}{x_{n+1}} = \frac{\theta_{n+1} - \theta_n}{\theta_n \theta_{n+1}} \]

\[ (18) \]

\[ \frac{x_k}{x_{k+1}} = \frac{\theta_{k+1} + 1}{\theta_k \theta_{k+1}} \]

\[ (19) \]

\[ \frac{x_k}{x_{k+1}} = \frac{(\theta_{k+1} + 1)(\theta_{k+1} - \theta_k)}{(\theta_k - \theta_{k+1})(\theta_k - \theta_{k+1})} \]

\[ (20) \]

Replace the results (17) (18) (19) into (14), we receive cascade controllers:

\[ R_k = \frac{\theta_{k+1} + 1}{\theta_k \theta_{k+1}} O_{i+1}^{-1} \]

\[ \ldots \]

\[ R_{n+1} \]

\[ (21) \]

From (21), we see that a controller at each loop depends only on the coefficients \( \theta_k \) of the previous loop, the current loop and the next loop, and just determine the value \( \theta_k \) we can determine the controller \( R_k \) (\( i = 1, n \)). If the dead time of the object in each loop is eliminated, then the structure of controllers is more simple and easy to use.

**Special case**:

If delay of any loop is much smaller than that of the next outer loop, the formulas (21) can be simplified as followings:

\[
\begin{align*}
R_k &= \frac{\theta_{k+1}}{\theta_k} O_{i+1}^{-1} \\
& \quad \ldots \\
R_{n+1} &= \frac{\theta_{n+1}}{\theta_n} O_{i+1}^{-1} \\
R_n &= \frac{1}{\theta_n} O_{i+1}^{-1}
\end{align*}
\]

We see that, to have the controller \( R_k \) (\( k = 1, n \)) just determine \( \theta_k \) and \( \theta_{k+1} \), and to have the controller \( R_n \) just determine \( \theta_n \).
IV. RETUNING THE CONTROL COEFFICIENTS BASED ON SOFT CHARACTERISTIC

Based on definition of the soft oscillation index in [8]:

\[
m(\omega) = m_0 \frac{1 - e^{-\tau \omega}}{\tau \omega}
\]

where: \( m_0 = \text{const}; \ \tau = \text{dead time of object}; \ \omega \geq 0 \) – the frequency. Hereafter, the symbol \( m \) is similar to \( m(\omega) \).

Replace \( s = -m \omega + j \omega, \ j = \sqrt{-1} \), into (8) we have the "soft characteristic" of \( k \)-th loop: \( H_k(-m \omega + j \omega) \) presented in Fig. 4.

Let \( H_k(-m \omega^* + j \omega^*) = P_k(\omega^*) + jQ_k(\omega^*), \ \Omega_k(\omega^*) = 0 \)

Consider relation with critical point: \((-1, j0), \) we have:

\[
K P_k(\omega^*) = -1 \Leftrightarrow K = -\frac{1}{P_k(\omega^*)} \quad (23)
\]

It's easy to see that in order to the soft characteristic goes through the point \((-1, j0), \) it is need only to multiply amplitude of the soft characteristic by \( K \).

Suppose that the controller of \( k \)-th loop is expressed in the form: \( R_k = K_k \frac{A(s)}{B(s)} \), where \( K_k \) – proportional coefficient.

Putting the form of \( R_k \) into (8), we receive the soft characteristic of \( k \)-th loop:

\[
H_k(s) = K_k \frac{A(s)}{B(s)} W_k(s) O_k(s) O_{1k}(s), \ s = -m \omega + j \omega
\]

So, in order to the soft characteristic goes through critical point, it is need only multiply proportional coefficient of controller \( R_k \) by correcting coefficient \( K \) defined by (23).

V. EXAMPLE

Let’s consider a cascade system with two loops (see Fig. 5). Where, the objects \( O_1 \) and \( O_2 \) have the following transfer functions:

\[
O_1(s) = \frac{e^{-0.32s}}{(1+0.9s)(1+0.38s)(1+0.22s)}
\]

\[
O_2(s) = \frac{e^{-0.035s}}{1+0.16s}
\]

It is obvious in the example the delay of the inner loop is much smaller than that of the outer loop. Therefore we can use the formulas (22). Eliminating dead time of the objects, we have the robust controllers:

\[
R_1 = \frac{1}{\theta_1 s} \frac{1}{1+0.9s+0.38s+0.22s}
\]

\[
R_2 = \frac{1}{\theta_2 s} \frac{1}{1+0.16s}
\]

We have:

\( \tau_1 = 0.320 \) minutes – the dead time of \( O_1, \)

\( \tau_2 = 0.035 \) minutes – the dead time of \( O_2. \)

Following [8], we choose oscillation indexes of inner and outer loops as: \( m_{c1} = 0.461, m_{c2} = 0.922. \) Then we calculate the lag-time coefficients of the loops [8]:

\[
\theta_1 = 1.348 \tau_1 = 1.348 * 0.320 = 0.43 \text{ minutes.}
\]

\[
\theta_2 = 1.943 \tau_2 = 1.943 * 0.035 = 0.068 \text{ minutes.}
\]

Consequently, the controllers are follows:

\[
R_1 = \frac{1+0.068s}{0.43s} \frac{1}{1+0.9s+0.38s+0.22s}
\]

\[
R_2 = \frac{1}{0.068s} \frac{1+14.65(1+0.16s)}{s}
\]

where: \( \Phi(s) = \frac{1}{(1+0.3T_{\text{min}}s)^3} \quad \text{a filter chosen so that the } \ R_k \ \text{will be proper; } T_{\text{min}} \ \text{the filter’s time constant chosen less or equal the smallest time constant of } \ R_1. \)
With the controllers, the step response of the system is shown in Fig. 6 (process a), and the soft characteristic of open outer loop shown in Fig. 7 (curve a).

![Figure 7](image)

Figure 7. The soft characteristic of the open outer loop.

According to simulation results, the step response has overshoot: $\delta = 40\%$, settling time: $T_s = 3.24$ minutes. The overshoot is still large and settling time is long.

In this case, the soft characteristic of open outer loop encompasses critical point. So we have to retune the proportional coefficient of controller $R_1$ in order to the soft characteristic is shrinked and crosses the critical point and by this condition the oscillation index ($m_c$) of the system will be guaranteed. After retuning we have:

$$R_1 = \frac{1.818(1+0.068s)(1+0.9s)(1+0.38s)(1+0.22s)}{s(1+0.022s)^2}$$

The $R_2$ is not need to change. Simulation results after the retuning are shown in Fig. 6 (process b) and Fig. 7 (curve b).

In last case the overshoot become: $\delta = 20.5\%$, and the settling time: $T_s = 1.92$ minutes.

The transfer function $R_1$ is rather complicated, but no problem if realize it by programmable microprocessor.

Below, we show other recipe to receive controller $R_1$ in mode of PID-structure for industry application.

Let $V_1$ is the equivalent object (Fig. 8).

![Figure 8](image)

Figure 8. The equivalent single loop structure of the cascade system

We have $V_1 = \frac{R_2O_2}{1 + R_2O_2}$. Approximating $V_1$ by model of second-order lag transfer function plus dead time (SOPDT) with using cleft-over-step algorithm of optimization [9], we receive the model:

$$MV_1 = e^{-0.528s} \left(1 + \frac{1}{0.667s}(1+0.667s)\right)$$

With model $MV_1$ after removing its dead time, we have:

$$MV_{1_{PR}} = \frac{1}{(1+0.667s)(1+0.667s)}$$

$$R_1 = \frac{1}{\theta_{PR} MV_{1_{PR}}} = \frac{1}{0.43s}(1+0.667s)(1+0.667s)$$

$$R_1 = 3.102 + \frac{1.326}{s} + 1.035s$$

Thus $R_1$ is the PID controller. Simulation result is shown in Fig. 6 (process c).

Retuning the proportional coefficient of the regulator by above methodology, we receive:

$$R_1 = 1.962 + \frac{1.437}{s} + 0.66s$$

Simulation result is shown in Fig. 6 (process d). The considered example also shows the capability of the cascade system to reject disturbance. Specifically, according to graphics in the Fig. 9, the disturbance response of cascade system almost decreases by 14 times comparing with single loop, from 0.421 to 0.029.

![Figure 9](image)

Figure 9. The disturbance response in cases of single loop and of two loops

VI. CONCLUSIONS

This article proposes a new method synthesizing controllers of cascade system based directly on the robust conditions applied to the system.

The proposed methodology of implementation is simple, easy to understand, easy to apply and is highly universal. It can be effectively applied to tune industrial control systems.

REFERENCES


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