A Robust Unscented Kalman Filter for Nonlinear Dynamical Systems with Colored Noise

M. Masoumnezhad, A. Jamali, and N. Nariman-zadeh
Department of Mechanical Engineering, Guilan University, Guilan, PO 3756 Iran
Email: {masoumnezhad, ali.jamali, nnzadeh}@guilan.ac.ir

***Abstract*—The Unscented Kalman filter (UKF) is the commonest filter for state estimation of a discrete-time nonlinear system corrupted with noise. If the process noise and measurement noise of nonlinear system are Gaussian and white, the UKF filter will be optimal. While the process or measurement noises are colored noise, the UKF filter will be instead suboptimal. In this paper, a robust UKF (R-UKF) filter is proposed in order to solve this problem. The proposed filter allows that the conventional UKF is applied for discrete-time nonlinear system corrupted with colored noise. Simulation results show that the R-UKF has better performance in estimation of the corrupted state of dynamical system by colored noise in comparison with those of using conventional UKF algorithm.**

***Index Terms*—unscented kalman filter, color noise, uncertainty, state estimation**

I. Introduction

In many actual applications, unknown quantities or defective measurements are recorded by observer [1]. One of the most popular filters to estimate those quantities in linear systems is Kalman filter [2]. The Kalman filter is a recursive filter that estimates the state of a linear dynamical system from a series of defective or corrupted data by noise and is widely used in the state estimation of many practical estimation problems [3].

When dynamical model is nonlinear, Ucented Kalman filter [4], [5] and Particle filter [6] are generally suggested by researchers. In special conditions, Kalman filter can be applied as an optimal filter, such as unscented Kalman filter, particle filter, etc. the process noise and measurement noise are exposed to Gaussian distribution and white noise. When process noise or measurement noise is colored noise (because it is correlated with itself at other time steps), classical Kalman filter is suboptimal [7]. To achieve optimal estimation by the use of classical Kalman filter, Kalman filter should be modified [8].

In linear systems, if the process or measurement noises is colored noise, it modify directly the state space equations and result equal equations but higher order system with white process noise [9].

In this paper, the modification of UKF algorithm for state estimation of a discrete-time nonlinear system is presented. The dynamical systems used in this approach are corrupted by colored noise. This approach is similar to linear colored-noise Kalman filter. Firstly, it is modified nonlinear state-space equations to higher order system with white noise; then, conventional UKF is applied for state estimation.

The organization of this paper is as follows, first, the conventional UKF algorithm for Gaussian and white noise is described; then, the contribution of this paper, Robust UKF algorithm for color process noise or measurement noise is derived. Finally, simulation results are presented that show comparisons between the conventional UKF algorithm with R-UKF algorithm to state estimation of discrete time dynamical systems subjected to uncertainty and color noise.

II. Conventional Unscented Kalman Filter

A discrete-time dynamical model of a nonlinear filtering problem is given as follows [10]:

$$x_{k+1} = f(x_k, u_k) + w_k \tag{1}$$

$$y_k = h(x_k) + v_k \tag{2}$$

where, x_k is state vector, y_k is measurement vector and w_k ,and v_k are process noise and measurement noise, respectively. Those noises are uncorrelated Gaussian white noises, $v_k = N(0, R)$, $w_k = N(0, Q)$.

The standard UKF algorithm includes 4 steps, which are given as follow [10]:

A. Initialization

$$\hat{x}_0 = E(x_0) \tag{3}$$

$$P_0 = E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] \tag{4}$$

where $\hat{x}_0$ and P_0 are initial state and initial error covariance, respectively. The superscript '*T*' denotes the matrix transpose and '*E*' refers to the expected value of a random variable.

B. Calculate Sigma-points

$$\chi_{0,k-1} = \hat{x}_{k-1} \tag{5}$$

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$$\chi_{i,k-1} = \hat{x}_{k-1} + (\sqrt{(L+\lambda)P_{k-1}})_i \quad (6)$$

$$\chi_{i+L,k-1} = \hat{x}_{k-1} - (\sqrt{(L+\lambda)P_{k-1}})_i \quad (7)$$

$$\omega_0^{(m)} = \frac{\lambda}{(L+\lambda)} \quad (8)$$

$$\omega_0^{(C)} = \frac{\lambda}{(L+\lambda)} + (1-\alpha^2+\beta^2) \quad (9)$$

$$\omega_i^{(m)} = \omega_i^{(C)} = \frac{1}{2(L+\lambda)} \quad (10)$$

$$\lambda = \alpha^2(L+\kappa) - L \quad (11)$$

where, $i=1,2,\ldots,L$ and L is the dimension of state vector. χ_i is sigma points in Unscented Transform that determined by the mean ($\hat{x}_k$) and covariance (P) of the priori random variable x_{k-1} . $\omega_i^{(m)}$ and $\omega_i^{(C)}$ are weighting parameters related to i^{th} sigma points. α , β and κ are designing coefficients which are selected by the designer [4].

C. Time-update Equations

$$\chi_{i,k|k-1} = f(\chi_{i,k|k-1}, u_k) \quad (12)$$

$$\hat{x}_k^- = \sum_{i=0}^{2L} \omega_i^{(m)} \chi_{i,k|k-1} \quad (13)$$

$$P_k^- = \sum_{i=0}^{2L} \omega_i^{(C)} [\chi_{i,k|k-1} - \hat{x}_k^-][\chi_{i,k|k-1} - \hat{x}_k^-]^T \quad (14)$$

$$\hat{y}_k^- = \sum_{i=0}^{2L} \omega_i^{(m)} h(\chi_{i,k|k-1}) \quad (15)$$

where, $\hat{x}_k^-$ and $\hat{y}_k^-$ are the priori estimated mean of x and y, respectively.

D. Measurement-update Equations

$$P_{k|k-1}^{yy} = \sum_{i=0}^{2L} \omega_i^{(C)} [h(\chi_{i,k|k-1}) - \hat{y}_k^-][h(\chi_{i,k|k-1}) - \hat{y}_k^-]^T \quad (16)$$

$$P_{k|k-1}^{xy} = \sum_{i=0}^{2L} \omega_i^{(C)} [\chi_{i,k|k-1} - \hat{x}_k^-][h(\chi_{i,k|k-1}) - \hat{y}_k^-]^T \quad (17)$$

$$K_k = (P_{k|k-1}^{xy})(P_{k|k-1}^{yy})^{-1} \quad (18)$$

$$\hat{x}_k = \hat{x}_k^- + K_k(y_k - \hat{y}_k^-) \quad (19)$$

$$P_k = P_k^- - K_k P_{k|k-1}^{yy} K_k^T \quad (20)$$

where, K_k is Kalman gain. $P_{k|k-1}^{yy}$ and $P_{k|k-1}^{xy}$ are the error covariance and the error cross covariance, respectively.

III. The Robust Unscented Kalman Filter for Color Noise

Defining a robust unscented Kalman filter which estimates state variables of a nonlinear system with colored noise is the main goal of this section. Consider the colored-noise dynamical model which is given as following:

$$x_{k+1} = f(x_k, u_k) + w_k \quad (21)$$

$$y_k = h(x_{k+1}) + v_k \quad (22)$$

where, w_k and v_k are the process noise and the measurement noise, respectively which are defined as follows [7]:

$$w_k = \Lambda_k w_{k-1} + \mu_k \quad (23)$$

$$v_k = \Gamma_k v_{k-1} + \varphi_k \quad (24)$$

where, Λ_k and Γ_k are constant matrices describing how the structure of the noise is propagated through time. μ_k and

φ_k are uncorrelated Gaussian white process and measurement noises, respectively. The covariance matrices of noises are $\varphi_k = N(0, R_k)$, $\mu_k = N(0, Q_k)$.

Because of the presence of colored noise in both the process noise (w_k) and measurement noise (v_k), the state space equations can be combined as follow:

$$\begin{bmatrix} x_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} f_1(x_k, u_k, w_k) \\ \Lambda_k w_k \end{bmatrix} + \begin{bmatrix} 0 \\ \mu_k \end{bmatrix} \quad (25)$$

where,

$$f_1(x_k, u_k, w_k) = f_k(x_k, u_k) + w_k \quad (26)$$

The modified state space equation can be defined as follow:

$$X_{k+1} = F(x_k, u_k, w_k) + \mathrm{M}_k \quad (27)$$

where, X_{k+1} is new state vector, F is new state space matrix and M_k is new process noise vector.

Also, to make the measurement white, auxiliary vector can be defined as follows:

$$Y_k = y_k - \Gamma_k y_{k-1} \quad (28)$$

$$Y_k = h_k(x_k) + v_k - \Gamma_k[h_{k-1}(x_{k-1}) + v_{k-1}] \quad (29)$$

$$Y_k = h_k(x_k) + \Gamma_k v_{k-1} + \varphi_k - \Gamma_k h_{k-1}(x_{k-1}) - \Gamma_k v_{k-1} \quad (30)$$

$$Y_k = h_k(x_k) - \Gamma_k h_{k-1}(x_{k-1}) + \varphi_k \quad (31)$$

If a new measurement matrix H_k is defined by the below equation, then

$$H_k(x_k) = h_k(x_k) - \Gamma_k h_{k-1}(x_{k-1}) \quad (32)$$

$$Y_k = H_k(x_k) + \varphi_k \quad (33)$$

where, Y_k is a new measurement vector that corrupted by Gaussian white noise φ_k .

Equations (27) and (33) are defined modified state space equations which are corrupted by uncorrelated Gaussian with white noises. The conventional UKF can be applied for modified state space equations to minimize the mean of the squared errors given by bellow equation [9].

$$MSE = \frac{\sqrt{norm(\hat{x}_k - x_k)}}{L} \tag{34}$$

where, x_k is the state vector which are corrupted by process noise given in equation (21) and $\hat{x}_k$ is the prediction of the state vector at time k which is given in equation (19).

Note, these modified equations which are given in equations (27) and (33) give back the conventional UKF equations when the operators Λ_k and Γ_k are set to zero.

IV. Simulation Results

In this section, two case studies have been considered to demonstrate the efficiency of the R-UKF algorithm in comparison with those of using the conventional UKF algorithm.

Example 1. Free falling body

Estimation of the altitude x_1, velocity x_2, and the ballistic coefficient x_3 for a body falling down the ground are considered. The dynamic equations are as follows [9]:

$$\dot{x}_1 = x_2 + w_{1,k} \tag{35}$$

$$\dot{x}_2 = \frac{\rho_0 \exp(-{x_1}/{D}) x_2{}^2 x_3}{2} - g + w_{2,k} \tag{36}$$

$$\dot{x}_3 = w_{3,k} \tag{37}$$

$$y(t_k) = \sqrt{M^2 + (x_1(t_k) - a)^2} + v_k \tag{38}$$

The process noise w_k and measurement noise v_k are uncorrelated Gaussian color noise. The noise model is given in equations (23) and (24). Λ_k and Γ_k are set to $\Lambda_k = \psi \times diag\{1,1,1\}$, and $\Gamma_k = \Phi$.

μ_k and φ_k are white Gaussian noise with covariance matrices chosen as $Q = 0.01 \times diag\{1,1,1\}$, *R=10000,* respectively. Initial conditions μ_{-1} and φ_{-1} are set to zero.

Also, the model constant parameters are considered as the following [9]:

$$\rho_0 = 2\,{lb.s^2}/{ft^4},\ D = 20000\,ft,\ g = 32.2\,{ft}/{s^2}, \tag{39}$$
$$M = 100000\,lb,\ a = 100000\,ft$$

The continues-time dynamical model has been discretized with the sampling period of 0.05 s. Then, discrete- time state space model is given as follows:

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \\ x_{3,k+1} \end{bmatrix} = \tag{40}$$
$$\begin{bmatrix} x_{2,k} \times dt + x_{1,k} \\ (\frac{\rho_0 \exp(-{x_{1,k}}/{D}) x_{2,k}{}^2 x_{3,k}}{2} - g) \times dt + x_{2,k} \\ x_{3,k} \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

The conventional UKF and the proposed method of this paper, R-UKF has been used to estimate the states of free falling body. Initial quantities of state vector and estimated error covariance are assumed

$$x_0 = \begin{bmatrix} 300000 & -20000 & 0.001 \end{bmatrix}^T,$$

$$P_0 = \begin{bmatrix} 1000000 & 0 & 0 \\ 0 & 4000000 & 0 \\ 0 & 0 & 10 \end{bmatrix} \tag{41}$$

The total simulation time is 15 *s*, all the experiment quantities are repeated for over 500 runs using Monte Carlo method. Two different algorithms, namely, conventional UKF and R-UKF are compared in several cases.

A. $\Psi = 0.8$ *and* $\Phi = 0$

In this section, there is only colored noise in the process noise, and the color content of the process noise is considered *0.8* .

Fig. 1 shows the magnitude of mean of squared error using conventional UKF is more than that of using R-UKF. The magnitude of mean of squared error given in equation (34) using R-UKF algorithm is 922.68, and using conventional UKF is 1227.28. In fact, the results of proposed method demonstrate the superiority of it.

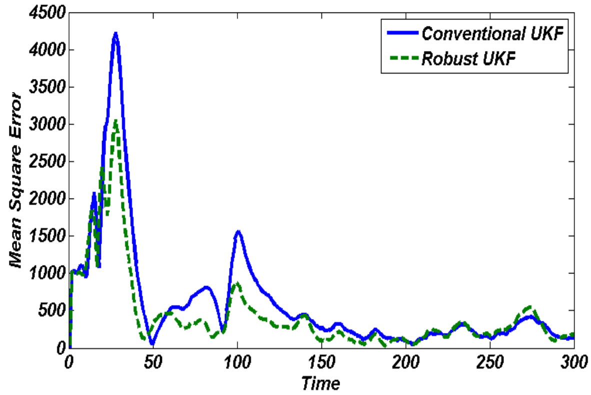

Figure 1. Mean square error using conventional UKF and R-UKF $\Psi = 0.8$ and $\Phi = 0$

B. $\Psi = 0.4$ *and* $\Phi = 0.2$

In this section, both process noise and measurement noise are colored noise. The results are illustrated in Figure 2. It is easy to know that the performance of R-UKF algorithm to estimate defective state variables with colored noise is better than conventional UKF algorithm.

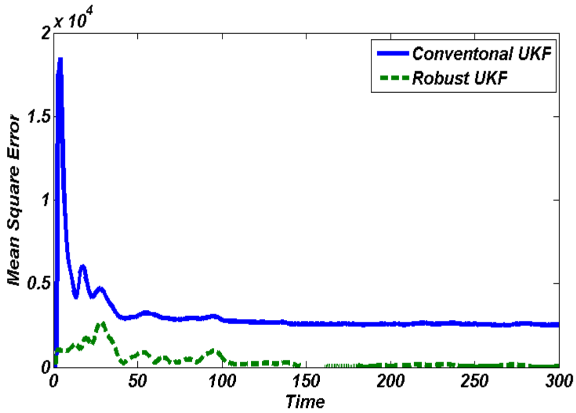

Figure 2. Mean square error using conventional UKF and R-UKF $\Psi = 0.4$ and $\Phi = 0.2$

The value of MSE using R-UKF algorithm is 875.24 and using conventional UKF algorithm is 4883.26.

C. Various Ψ and Φ

In this section, two different methods (R-UKF and conventional UKF) using many cases of color content of process noise or measurement noise is compared together.

Simulation results are shown in table 1. Table 1 shows that when color content of process noise and measurement noise is set to zero, $\psi = \Phi = 0$. The mean of squared error using R-UKF is equal to that of using conventional UKF and noise is white. As Ψ or Φ increases, the color of the process noise or measurement noise increases. The magnitude of mean of squared error using of robust UKF (R-UKF) algorithm is less than conventional UKF, and R-UKF algorithm have better performance to estimate state variables corrupted by color noise compared with conventional UKF algorithm.

Moreover, it can be noticed from Table I (when Ψ is kept constant with variable Φ), that the use of R-UKF filter on the systems with color measurement noise may have significant effect on the state estimation than that of using UKF filter. It is, therefore, recommended to use such R-UKF filter to those systems in order to obtain better state estimations.

TABLE I. THE MEAN SQUARE ERROR USING CONVENTIONAL UKF AND R-UKF FOR A FREE FALLING BODY

Ψ	Φ	Conventional UKF	Robust UKF
0	0	865.91	866.73
0.5	0	976.47	879.64
0.9	0	1421.85	1030.48
0.5	0.3	5265.95	879.69
0.5	0.7	6749.82	879.69
0.9	0.3	5376.75	1031.52
0.9	0.7	6850.48	1031.55

Example 2. Frequency modulated signal

Estimation of state variables for a frequency modulated signal model is considered for the second case study [11]. The model equation can be described by:

$$x_{1,k+1} = \mu x_{1,k} + w_{1,k} \tag{42}$$

$$x_{2,k+1} = \arctan(\lambda x_{2,k} + x_{1,k}) + w_{2,k} \tag{43}$$

$$y_{1,k} = \cos(x_{2,k}) + v_{1,k} \tag{44}$$

$$y_{2,k} = \sin(x_{2,k}) + v_{2,k} \tag{45}$$

where, the constant parameters are $\mu = 0.9$ and $\lambda = 0.99$. The process noise w_k and measurement noise v_k are Gaussian color noise. The noise model is assumed the same as the free falling body problem and, Λ_k and Γ_k are set to $\Lambda_k = \psi \times diag\{1,1\}$, and $\Gamma_k = \Phi \times diag\{1,1\}$, respectively.

Also, μ_k and φ_k are Gaussian and white noise with covariance matrices chosen as $Q = 0.04 \times diag\{1,1\}$, $R = 0.1 \times diag\{1,1\}$, respectively.

Two different algorithms, namely, conventional UKF and R-UKF have been used to estimate the states of frequency modulated signal. Initial quantities of state vector and estimated error covariance are assumed, as follows:

$$x_0 = [1 \quad 1]^T,\ P_0 = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \tag{46}$$

In the simulation, all the experiments are calculated by a Monte-Carlo approach using 500 randomly drawn samples. Many cases have been considered to demonstrate the performance of proposed algorithm in comparison with conventional UKF algorithm.

A. $\Psi = 0.8$ and $\Phi = 0$

In this section, it is considered that process noise is only colored noise and measurement noise is white noise. The performance is characterized by mean of squared error given in equation (34) over 500 Monte-Carlo runs. Figure 3 is shown that robust UKF algorithm perform better than the conventional UKF algorithm. The mean of squared error of R-UKF is 0.0462 while it is 0.0551 for the conventional UKF.

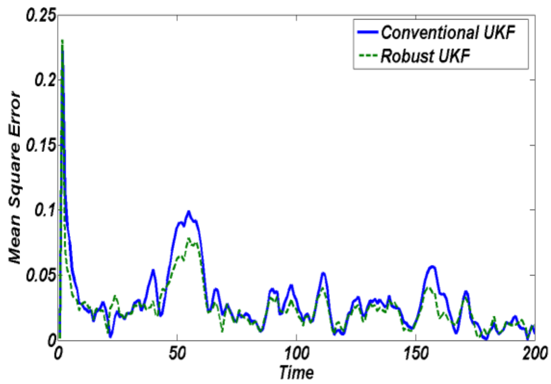

Figure 3. Mean square error using conventional UKF and R-UKF $\Psi = 0.8$ and $\Phi = 0$

B. $\Psi = 0.4$ *and* $\Phi = 0.2$

In this section, both process and measurement noises have been assumed colored noise. As can be seen form Figure 4, the robust UKF algorithm generates superior performance in mean of squared error than the conventional UKF algorithm. The mean of squared error using R-UKF algorithm is calculated 0.0381 and it is 0.0508 using conventional UKF algorithm.

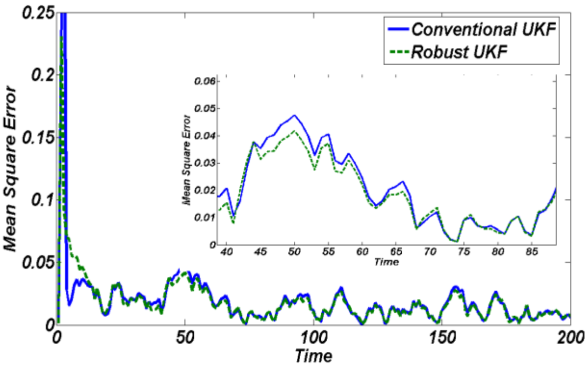

Figure 4. Mean square error using conventional UKF and R-UKF $\Psi = 0.4$ and $\Phi = 0.2$

C. *Various* Ψ *and* Φ

In this section, the several values of Ψ and Φ have been used to simulate correlated noises. Table II shows the mean of squared error for the conventional UKF and robust UKF (when the colored noise is used).

When correlation of noises is set to zero, the mean of squared error is same for the two filters, as expected. However, when process noise or measurement noise is colored noise, the robust UKF algorithm performs noticeably better than the conventional UKF algorithm.

Also, it can be observed from Table II, that the mean of squared error has many different values for conventional UKF filter (when Ψ is kept constant with variable Φ), while that is nearly constant for R-UKF filter.

TABLE II. THE MEAN SQUARE ERROR USING CONVENTIONAL UKF AND R-UKF FOR A FREQUENCY MODULATED SIGNAL MODEL

Ψ	Φ	Conventional UKF	Robust UKF
0	0	0.0359	0.0360
0.5	0	0.0418	0.0390
0.9	0	0.0659	0.0516
0.5	0.3	0.0534	0.0391
0.5	0.7	0.0564	0.0391
0.9	0.3	0.0757	0.0516
0.9	0.7	0.0783	0.0517

V. CONCLUSION

When the process noise or measurement noise is colored noise, UKF algorithm is suboptimal for state estimation. In this paper, a robust UKF algorithm has been proposed to state estimation of discrete time dynamical systems subjected to uncertainty and color noise.

In robust UKF algorithm, the state space equations of nonlinear dynamical system corrupted by color noise are changed to state space equations with white noise. The conventional UKF algorithm is then applied to the modified state space equations.

The simulation results show that the R-UKF algorithm is more robust to the colored noise uncertainty than the conventional UKF. The R-UKF algorithm has better performance than conventional UKF to estimate the observed and unobserved states of system, particularly, when the color measurement noise is more persistent than the process noise.

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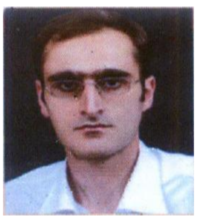
M. Masoumnezhad was born in Rasht,Iran in 1980, and received his B.S. in Fluid Mechanical Engineering from Guilan Univercity, Iran, in 2002 and M.Sc. (Eng.) in Mechanical Engineering from Guilan univercity, Iran, in 2004.

Mojtaba Masoumnezhad Ph.D candidate at guian university and his research interest covers intelligent control and modeling using evolutionary algorithms, and GMDH-type neural networks, multi-objective optimization in modelling and control, robust and reliabilty-based design and optimization using evolutionary algorithms.



A. Jamali received his B.S. in Solid Mechanical Engineering from Guilan Univercity, Iran, in 2002 and M.Sc. (Eng.) in Mechanical Engineering from Iran University of Science and Technology, Iran, in 2004, and the Ph.D. degree in Solid Mechanical Engineering from Guilan Univercity, Iran, in 2009.

Ali Jamali is currently a assistant professor of Control Engineering in Guilan University. His research areas include intelligent control and modeling using evolutionary algorithms, fuzzy logic, and GMDH-type neural networks, multi-objective optimization in modelling and control, robust and reliabilty-based design and optimization using evolutionary algorithms.

N. Nariman-zadeh received his M.Sc. (Eng.) in Mechanical Engineering from Tehran University, Iran, in 1986 and the Ph.D. degree in Intelligent control from Salford University, England, in 1996. He then joined the Department of Industrial Manufacturing and Systems Engineering of the University of Hong Kong for his postdoctoral period.

N. Nariman-zadeh professor in the Department of Mechanical Engineering of the University of Guilan and Intelligent-based Experimental Mechanics Center of Excellence, School of Mechanical Engineering, Faculty of Engineering, University of Tehran. He was also a visiting Associate Professor at the University of Birmingham, UK, 2002-2003. He has served as the Head of Mechanical Eng. Dept., Dean of Faculty of Engineering, Deputy of Chancellor in Research of the University of Guilan, Reviewer of the Journal of Engineering Application of Artificial Intelligence, Journal of Applied Mathematical Modeling, Journal of Materials Processing Technology, Journal of Engineering Manufacture, International Journal of Systems Science, Journal of Automobile Engineering, and Editorial Board of Open Mechanical Engineering. His research interests are intelligent control and modeling using evolutionary algorithms, fuzzy logic, and GMDH-type neural networks, multi-objective optimization in modelling and control, robust and reliabilty-based design and optimization using evolutionary algorithms.