The Structure of Common Linear Copositive Lyapunov Functions for Continuous time Switched Positive Linear Systems

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Abstract—This paper addressed the structure of common linear copositive Lyapunov function (CLCLFs) for continuous-time switched positive linear systems (CSPLS). In this note, first of all, for the \( n \times n \)-dimensional matrices family, which entries are the Hurwitz and Metzler switch matrices of an \( n \)-dimensional CSPLS, we presented, by a straightforward algebraic computation, a procedure for constructing a group \( n \)-dimensional vectors, further, we provided a structure theorem of CLCLFs, which shown that all CLCLFs of a CSPLS make a positive open cone generated by these special selected vectors, finally, an numerical example is worked out to illustrate our main results. The results of this technical note are extensions of some recent results in Ref. [1].

Index Terms—positive switched system, common linear copositive Lyapunov function, structure

I. INTRODUCTION

A continuous-time switched positive system (CSPS) is a type of hybrid dynamic system that consists of a family of positive state-space models (Ref. [2], [3]) and a switching signal, which determines the switching between sub-systems. In recent years, CSPSs have been received much attention due to their broad applications in communication systems (Ref. [4]), formation flying (Ref. [5]) and other areas.

In the theory of CSPSs, the stability problem is investigated extensively by many researchers (Ref. [6]-[13]), especially for the stability under arbitrary switching.

We particularly mention a recent paper by Fornasini and Valcher, Ref. [13], where a complete characterization for the existence of a CLCLF for a set of \( nxn \)-dimensional Metzler matrices \( A_n = \{A_1, A_2, \ldots, A_n\} \) it usually requires to check \( m^n \) matrices to be Hurwitz in order to guarantee the existence of a CLCLF for a set of Metzer matrices, \( A_n \). Although some necessary and sufficient conditions have been given in Ref. [13], it seems to be difficult to verify those conditions for sufficiently large \( m \) and \( n \).

Zhaorong Wu and Yuangong Sun show in Ref. [1] that the existence of a CLCLF for a family of Metzler matrices \( A_n \) can only be determined by \( n-1 \) Metzler Hurwitz matrices, instead of \( m^n \) matrices, the existence of a CLCLF for the family \( A_n \) is equivalent to the existence of \( n-1 \)characteristic constants less than 1, an easily verifiable computation algorithm is provided, which can be carried out by solving the inverse of some matrices rather than checking \( m^n \) Metzler matrices to be Hurwitz. Unfortunately, the structure of CLCLFs has not discussed in all known literature. For example, if there exist CLCLFs for the family \( A_n \), how to structure these CLCLFs? In this note, we will answer this problem.

The main contribution of this note is two folds. We not only shown that all CLCLFs make a positive open cone generated by \( n \) special selected vectors, but also gives a method to choose these \( n \) vectors.

This note is organized as follows. Problem statements and preliminaries are given in Section II. Section III gives the main results. Finally, an illustrating example is presented in IV.

Notations: For any positive integer \( m \), \( (m) \) is the set of integers \( \{1, 2, \ldots, m\} \). \( I_n \) is an \( n \)-dimensional identity matrix. A diagonal matrix is denoted by diag\{...\}. Say \( A > 0 \) if all entries of matrix \( A \) are positive (nonpositive, negative). For a set \( \Omega \) of \( n \)-dimensional vectors, \( \phi \) is said to be a maximal vector if \( \phi \in \Omega \) and \( \phi \leq \psi \) for any \( \psi \in \Omega \). A Metzler matrix is a real square matrix, whose off-diagonal entries are nonnegative. A Hurwitz matrix is a square matrix, if the real part of each eigenvalue of this matrix is negative. The \( j \)-th column and the \( (i, j) \)-entry of a matrix \( A_k \) is denoted by col\(\{A_k\}\) and \( a_{ij}^{(k)} \), respectively. For any \( j \in \Omega \), \( e_j \) is an \( n \)-dimensional unit vector whose \( j \)-th entry is 1.

II. PROBLEM STATEMENTS

Consider the following continuous-time switched positive linear system (CSPLS)

\[
\dot{x}(t) = A_{\sigma(t)}x(t), \quad t \geq 0
\]

where \( x \) is the \( n \)-dimensional state vector with \( n \geq 2 \), the piecewise continuous function \( \sigma: [0, +\infty) \rightarrow \Omega \) is the switching signal, \( A_k, k \in \Omega \), are Metzler and Hurwitz matrices. All these matrices make a family denoted by...
Assume \( A_\kappa = \{ A_\kappa : k \in \langle m \rangle \} \)

It is well known that, if \( x(t), t \in [0, + \infty) \), is a solution of system (1) satisfying \( x(0) \geq 0 \), then \( x(t) \geq 0 \) for any \( t \in [0, + \infty) \). System (1) is called to be global asymptotic stability if each solution with the initial condition \( x(0) \geq 0 \) is asymptotically stable under arbitrary switching.

When considering the stability of system (1) under arbitrary switching, it is usual to find a CLCLF for system (1). By a CLCLF for system (1) (or the family \( A_\kappa \)) we mean a function \( V(x) = v^T x \) satisfying \( v > 0 \) and \( v^T A_\kappa < 0 \) for each \( k \in \langle m \rangle \). For a family \( A_\kappa \) of \( n \times n \)-dimensional Metzler Hurwitz matrices, we have that \( v^T A_\kappa < 0 \) is equivalent to \( v^T (A_\kappa D_\kappa^{-1}) < 0 \), where \( D_\kappa = \text{diag} (a_{11}, a_{22}, \ldots, a_{nn}) \) and \( v \) is an \( n \)-dimensional vector. Without loss of generality, throughout this note we assume that

(H): \( A_\kappa \) is a family of Metzler Hurwitz matrices, and \( a_{ij} \leq -1 \) for \( i \in \langle n \rangle \) and \( k \in \langle m \rangle \).

The following lemma is one of necessary and sufficient conditions for the existence of a CLCLF for system (1) given

**Lemma 1**: Given a family \( A_\kappa \) of \( n \times n \)-dimensional Metzler Hurwitz matrices, there exists a CLCLF for the family \( A_\kappa \) if and only if the following Metzler matrix

\[
A_{k_1, \ldots, k_n} = \left[ \text{col}_i (A_{k_1}), \ldots, \text{col}_n (A_{k_n}) \right]
\]

is a Hurwitz matrix for all possible values of \( k_1, \ldots, k_n \in \langle n \rangle \).

In the following, for \( p \in \langle n \rangle \), let

\[
A_{k_1, \ldots, k_n}^{p} = E_p A_{k_1, \ldots, k_n} E_p^T
\]

where \( k; k_1, \ldots, k_n \in \langle m \rangle \), \( E_p = \{ 1, \ldots, p \} \) is an \( p \times n \)-dimensional matrix for \( p < n \) and \( E_n = I_n \). Denote

\[
A_p = \{ A_p^{i} : k_1, \ldots, k_n \in \langle m \rangle \}, p \in \langle n \rangle
\]

Decompose \( A^{p+1}_{k_1, \ldots, k_{n+1}} \) as the following:

\[
A^{p+1}_{k_1, \ldots, k_{n+1}} = \begin{bmatrix}
A^{p}_{k_1, \ldots, k_n}, \\
\xi_{p+1}, \\
\eta_{p+1}
\end{bmatrix}
\]

where \( \xi_{p+1} \) and \( \eta_{p+1} \) are the \( p \)-dimensional vectors whose \( i \)-th entries \((p \geq i \geq 1)\) are the \( (p+1), i \)-th entry and the \( (i, p+1) \)-th entry of the matrix \( A^{p+1}_{k_1, \ldots, k_{n+1}} \), respectively.

Denote

\[
T_p = \{ \theta_{k_1, \ldots, k_n} : k_1, \ldots, k_n \in \langle m \rangle \}, p \in \langle n-1 \rangle
\]

where \( \theta_{k_1, \ldots, k_n} = (\theta_{1, k_1, \ldots, k_n}, \ldots, \theta_{n, k_1, \ldots, k_n}) \) satisfy

\[
\theta_{k_1, \ldots, k_n}^{T} A_{k_1, \ldots, k_n}^{p} = -\eta_{p+1}^{T}.
\]

Before establishing the main result, we first present the following lemmas given in Ref. [1], which play a key role in proof of the main results.

**Lemma 2** (Ref. [1]) For given \( p \in \langle n-1 \rangle \), assume that there exists a CLCLF for the family \( A_p \). Then, there exists a \( p \)-tuple \((k_1, \ldots, k_{n-p})\) with \( k_{n-p} \in \langle m \rangle \) for each \( i \in \langle p \rangle \), such that \( \theta_{k_1, \ldots, k_{n-p}} \leq \theta_{k_1, \ldots, k_{n-p}}^{T} A_{k_1, \ldots, k_{n-p}} \leq -\eta_{n-p+1} \) with \( k_{n-p} \in \langle m \rangle \).

**Lemma 3** (Ref. [1]) For given \( p \in \langle n-1 \rangle \), assume that there exists a CLCLF for the family \( A_p \). Then, there exists a \( p \)-tuple \((k_1, \ldots, k_{n-p})\) with \( k_{n-p} \in \langle m \rangle \) for each \( i \in \langle p \rangle \), such that \( \theta_{k_1, \ldots, k_{n-p}} \leq \theta_{k_1, \ldots, k_{n-p}}^{T} A_{k_1, \ldots, k_{n-p}} \leq -\eta_{n-p+1} \) with \( k_{n-p} \in \langle m \rangle \).

In the following lemma, we see that the characteristic constants \( \lambda_p \), \( p \in \langle n-1 \rangle \) play a key role in the existence of a CLCLF.

**Lemma 4** (Ref. [1]) There exists a CLCLF for the family \( A_p \) if and only if \( \lambda_p < 1 \).

### III. Main Results

In this section, we assume that there exists a CLCLF for the family \( A_\kappa \) of \( n \times n \)-dimensional metzler and Hurwitz matrices. Our main task is to find the methods how to construct these CLCLFs.

If \( v^{(n)} = (x_1, x_2, \ldots, x_n)^T \) is a CLCLF of the family \( A_\kappa \), then \( v^{(n)} = (x_1, x_2, \ldots, x_{n-1})^T \) is easily to see that \( v^{(n)} \) is a CLCLF of the family \( A_{n-1} \). From Lemma 2, for \( p = n-1 \), there exists \((n-1)\)-tuple \((k_1, \ldots, k_{n-1}) \) with \( k_{n-1} \in \langle m \rangle \) such that

\[
\theta_{n-1}^{(n)} \| V^{(n)} \| = -\eta_{n-1}^{(n)}
\]

where

\[
\theta_{n-1}^{(n)} = \theta_{k_1, \ldots, k_{n-1}} \quad \eta_{n-1}^{(n)} = \eta_{k_1, \ldots, k_{n-1}}
\]

If the \( (n-1) \)-dimensional vector

\[
\theta_{n-1}^{(n)} = (w_1, w_2, \ldots, w_{(n-1)n})^T
\]

set the \( n \)-dimensional vector

\[
Q_n = (w_1, w_2, \ldots, w_{(n-1)n}, 1)^T
\]

From the equation (5), for \( p = n-1 \), there exists \( k_{n-1} \in \langle m \rangle \) such that
\[
\max\{\theta_{n-1}^{(s)} \xi_k : k \in (m)\} = \theta_{n-1}^{(s)} \xi_{k_{n-1}}^T,
\]
which denoted by \(\lambda_n^{(s)}\), that is
\[
\lambda_n^{(s)} = \theta_{n-1}^{(s)} \xi_{k_{n-1}}^T \tag{7}
\]
By lemma 3, we see \(\lambda_n^{(s)} < 1\). Set
\[
U = A_{\{k_{n-1}, k, \ldots, k_1\}} A_{\{k_{n-1}, k_1, \ldots, k\}}^{-1}
\]
A straightforward computation yields that
\[
\Omega_{n}^T U = -(1-\lambda_n^{(s)}) e_n \tag{8}
\]
In essence, \(\Omega_n\) is the solution of the equation
\[
\Omega_n^T U(n)^{o} = 0
\]
where \(U(n)^{o}\) is the matrix constructed by removing the \(n\)th column of the matrix \(U\). In the following, according to the similar way, we will structure a group of \(n\)-dimensional vectors \(\{\Omega_1, \Omega_2, \ldots, \Omega_n\}\), and final we show that all CLCLFs of the family \(A_n\) can be cetained by these vectors. To this end, for each \(j \in (n)\), set \(U(j)^{o}\) is the \(n \times (n-1)\)-dimensional matrix constructed by removing the \(j\)th column of the matrix \(U\) and
\[
Q_j = (w_{j,1}, \ldots, w_{j,(j-1)}, w_{(j+1),1}, \ldots, w_{n})^T \tag{9}
\]
is the \(n\)-dimensional vector satisfying
\[
\Omega_j^T U(j)^{o} = 0 \tag{10}
\]
**Theorem 1**: \(v_n\) is a CLCLF of the family \(A_n\) if and only if \(v_n = \sum_{j=1}^{n} v_j \Omega_j\), \(\{v_1, v_2, \ldots, v_n\}\) is a group of positive constants.

**Proof**: For \(j \in (n)\), \(k, k_1, k_2, \ldots, k_n \in (m)\), assume \(U_{k,j}^{(k)}\) and \(U_{k_1,j}^{(k_1)}\) are the \(n(1)\)-dimensional matrices constructed by removing the \(j\)th column of the matrices \(A_k\) and \(A_{k_1}\), respectively; \(\eta_k^{(j)}, \eta_{k_1}^{(j)}\) and \(\eta_{k_1,k_2}^{(j)}\) are the \((j)\)-row of the matrices \(U_{k,j}^{(k)}\) and \(U_{k_1,j}^{(k_1)}\), respectively; \(\xi_k^{(j)}\) and \(\xi_{k_1,k}^{(j)}\) are the \((n-1)\)-dimensional vectors constructed by removing the \(j\)th entry of \(\text{col}(A_k)\) and \(\text{col}(U)\), respectively; Further, \(V_{k,j}^{(k)}\), \(V_{k_1,j}^{(k_1)}\) and \(V_{k_1,k_2}^{(j)}\) are the matrices constructed by removing the \(j\)th row of the matrices \(U_{k,j}^{(k)}\), \(U_{k_1,j}^{(k_1)}\) and \(U_{k_1,k_2}^{(j)}\), respectively. We further set
\[
V_{n-1}^{(0)} = (V_1^{(0)}, V_2^{(0)}, \ldots, V_{n}^{(0)})
\]
If \(v = (x_1, x_2, \ldots, x_n)^T\) is a CLCLF of the family \(A_n\), then \(v_{n-1}^{(0)} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)^T\) is a CLCLF of \(V_{n-1}^{(0)}\). By Lemma 2, there exists \(\theta_{n-1}^{(s)}\) such that
\[
[\theta_{n-1}^{(s)}] \xi_{k_{n-1}}^T \leq -[\eta_{k_{n-1}}^{(s)}] \xi_{k_{n-1}}^T, \leq -[\eta_{k_{n-1}}^{(s)}] \xi_{k_{n-1}}^T \tag{11}
\]
for all \(k, k_1, k_2, \ldots, k_n \in (m)\), and if
\[
\theta^T V_{k_{n-1}, k_1}^{(s)} = [\theta_{n-1}^{(s)}] \xi_{k_{n-1}}^T \tag{12}
\]
then
\[
\theta \leq \theta_{n-1}^{(s)} \tag{13}
\]
As \(U\) is Hurwitz matrix, then for any \(j \in (n)\), \(V_j^{(0)}\) is also a Hurwitz matrix, so there exists an unique solution \(\Omega_j\) for the equation (10) (11). Further from (12) (13) and (14) we get that the vector
\[
w_j = (w_{j,1}, \ldots, w_{j,(j-1)}, w_{(j+1),1}, \ldots, w_{n})^T
\]
satisfying
\[
w_j \leq \theta_{n-1}^{(s)} \tag{15}
\]
A straightforward computation yields that
\[
\Omega_j^T U = -(1-\lambda_n^{(s)}) e_j, j \in (n) \tag{16}
\]
where
\[
\lambda_n^{(s)} = w_j^T \xi_{k_{n-1}}^T \tag{17}
\]
by Lemma 3 we see
\[
\max\{\theta_{n-1}^{(s)} \xi_{k_{n-1}}^T : k \in (m)\} < 1 \tag{18}
\]
To sum up (15) (16) (17) (18) we get that
\[
\lambda_n^{(s)} < 1, \quad j \in (n) \tag{19}
\]
Now we prove the necessity and sufficiency, respectively.

**Necessity.** If \(v\) is a CLCLF of the family \(A_n\), we know from lemma 1 that \(U\) is a Hurwitz matrix and \(v^T U < 0\). Set
\[
v^T U = -(c_1, c_2, \ldots, c_n), c_j > 0, j \in (n) \tag{20}
\]
and set constants as follows \(v_j = \frac{c_j}{1-\lambda_n^{(s)}}\), \(j \in (n)\), by (19)
we see for any \(j \in (n)\), \(v_j > 0\). Further by (16)(20) we see
\[
v^T U = -\sum_{j=1}^{n} v_j(1-\lambda_n^{(s)}) e_j = \sum_{j=1}^{n} v_j \Omega_j^T U \quad \text{thus}
\]
\[
(v^T - \sum_{j=1}^{n} \Omega_j^T) U = 0 \quad \text{by} \ U \text{ is a Hurwitz family, then}
\]
\[
v = \sum_{j=1}^{n} \Omega_j.
\]

**Sufficiency.** For any group of positive constants \(\{v_1, v_2, \ldots, v_n\}\), let \(v = \sum_{j=1}^{n} \Omega_j\), we prove \(v\) is a CLCLF of the family \(A_n\). From (12) (13) (14), we get that for any \(k \in (m)\), \(\Omega_j^T A_k = (w_{j_1}, w_{j_2}, \ldots, w_{j_n})^T, k \in (n)\), where
Then 

\[ w_{jk}^{(k)} = W_{jk}^T e_{jk}(j) - 1 , \quad \text{and} \quad w_{jk}^{(k)} \leq 0, i \in \{n\}, \quad i \neq j, \]

\[ \Omega_j^T A_k \leq W_{jk}^T \epsilon_j, j \in \{n\}. \]

From (15), (18) we get that 

\[ W_{jk}^T e_{jk}^{(j)}(j) , \quad \text{so} \quad W_{jk}^T e_{jk}^{(j)} - 1 < 0. \]

Further from (17), (18), (19), we have that 

\[ v_i^T A_k = \sum_{j=1}^{n} v_i \Omega_j^T A_k \leq \sum_{j=1}^{n} v_i w_{jk}^{(k)} \epsilon_j. \]

Thus \( v_i^T A_k \geq 0, \) namely \( v \) is CLCLF of the family \( A_n. \)

The proof of Theorem 1 is complete.

IV. NUMERICAL EXAMPLE

A numerical example is worked out to illustrate the main results in this section.

Let \( A_j = \{ A_1, A_2 \} (\text{Ref. [14]}), \) where

\[ A = \left[ \begin{array}{ccc} 1 & 1 & \frac{1}{19} \\ \frac{1}{19} & -1 & \frac{1}{2} \\ 1 & 1 & -1 \end{array} \right], \quad A_i = \left[ \begin{array}{ccc} -1 & \frac{1}{19} & \frac{1}{2} \\ \frac{1}{19} & -1 & \frac{1}{2} \\ 2 & 2 & -1 \end{array} \right] \]

From Ref. [14], there exist CLCLFs for the family \( A_j, \)

\[ U = A_{1,1} = \left[ \begin{array}{cc} -1 & 1 \\ \frac{1}{19} & -1 \\ 2 & 1 \end{array} \right] \]

Then \( U^{(1)}, U^{(2)}, U^{(3)} \) is, respectively

\[ \begin{array}{ccc} 1 & \frac{1}{19} & \frac{1}{2} \\ \frac{1}{19} & -1 & \frac{1}{2} \\ 2 & 1 & -1 \end{array} \]

\[ \begin{array}{ccc} -1 & 1 \\ \frac{1}{19} & -1 \\ 2 & -1 \end{array} \]

\[ \begin{array}{ccc} -1 & 1 \\ \frac{1}{19} & -1 \\ 2 & 1 \end{array} \]

From \( \Omega_j^T U^{(j)} = 0 \) \( j \in \{3, \} \), we get \( \Omega_1^T = (1, \frac{1}{19}, \frac{1}{2}) \), \( \Omega_2^T = (\frac{2}{19}, 1, \frac{1}{2}) \), \( \Omega_3^T = (8.9, 1). \) Then, by theorem 1, \( v \) is a CLCLF of the family \( A_j \), if and only if there is a group of positive constants \( v_1, v_2 \) and \( v_3 \) such that

\[ v = v_1 \Omega_1 + v_2 \Omega_2 + v_3 \Omega_3. \]

Particularly, set \( v_1 = 2.7, v_2 = 2.55, v_3 = 0.45, \) then \( v = (8.5, 9.6, 1) \) is a CLCLF of the family \( A_j \) which is consistent with Ref. [1].

ACKNOWLEDGMENT

The authors wish to thank the support of the national natural science foundation of China under grant nos 60704039 and 61174217.

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